Finite Automata

Part Two

Recap from Last Time

DFAs

- A ***DFA*** is a
  - ***D***eterministic
  - ***F***inite
  - ***A***utomaton
- DFAs are the simplest type of automaton that we will see in this course.

DFAs

- A DFA is defined relative to some alphabet Σ.
- For each state in the DFA, there must be ***exactly one*** transition defined for each symbol in Σ.
  - This is the “deterministic” part of DFA.
- There is a unique start state.
- There are zero or more accepting states.

If D is a DFA, the ***language of D***, denoted $\mathscr{L}(D)$, is $\{ w \in \Sigma^* \mid D \text{ accepts } w \}$.

A language L is called a ***regular language*** if there exists a DFA D such that $\mathscr{L}(D) = L$.

NFAs

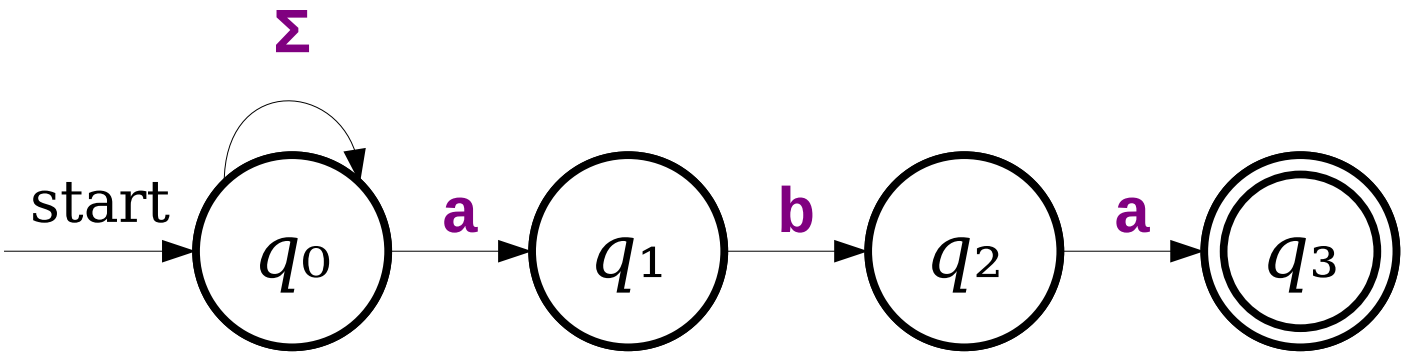
- An ***NFA*** is a
  - ***N***ondeterministic
  - ***F***inite
  - ***A***utomaton
- Can have missing transitions or multiple transitions defined on the same input symbol.
- Accepts if *any possible series of choices* leads to an accepting state.

New Stuff!

Intuiting Nondeterminism

- Nondeterministic machines are a serious departure from physical computers. How can we build up an intuition for them?
- There are two particularly useful frameworks for interpreting nondeterminism:
  - ***Perfect positive guessing***
  - ***Massive parallelism***

Perfect Positive Guessing



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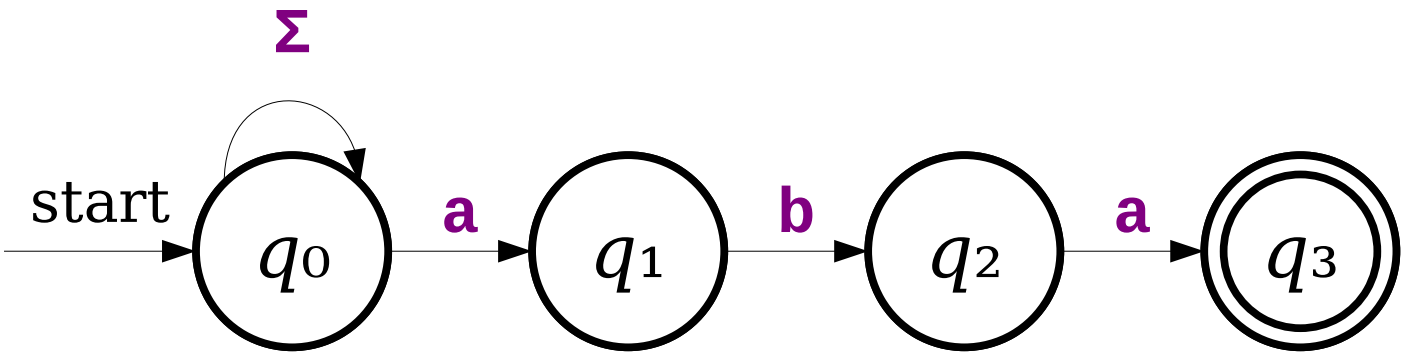

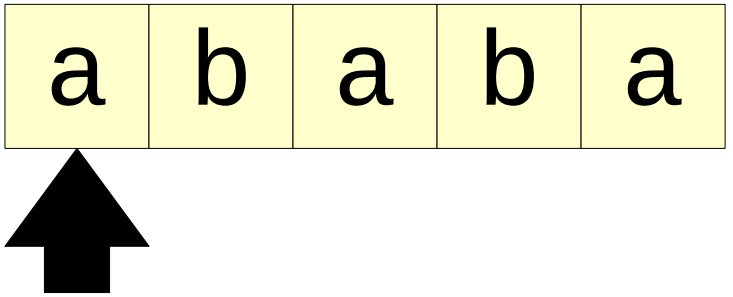

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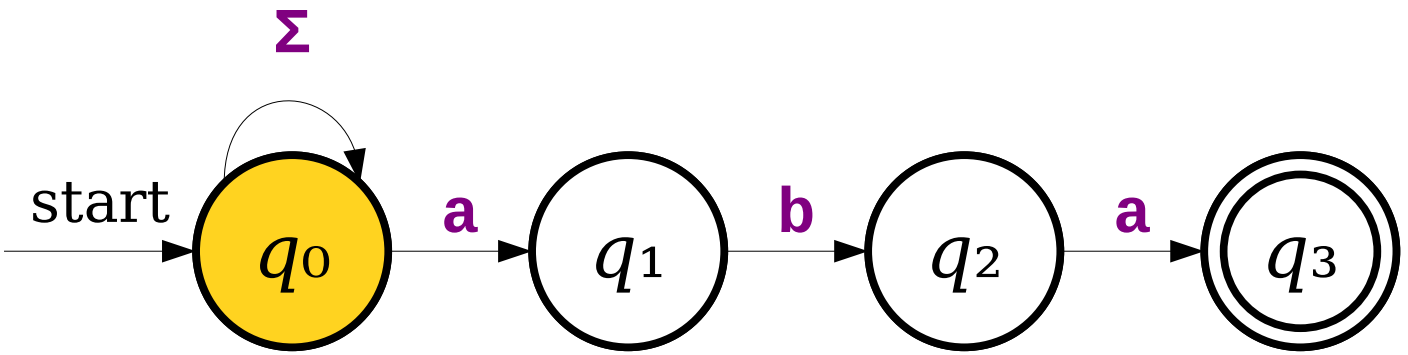

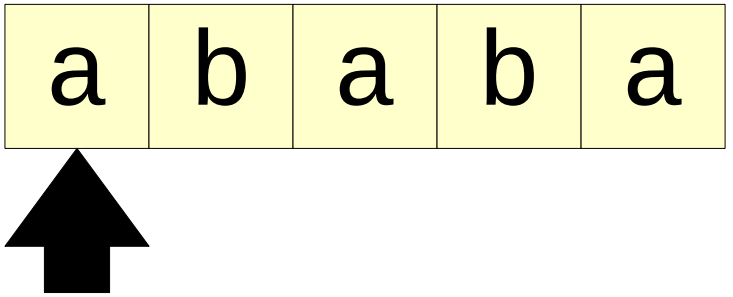

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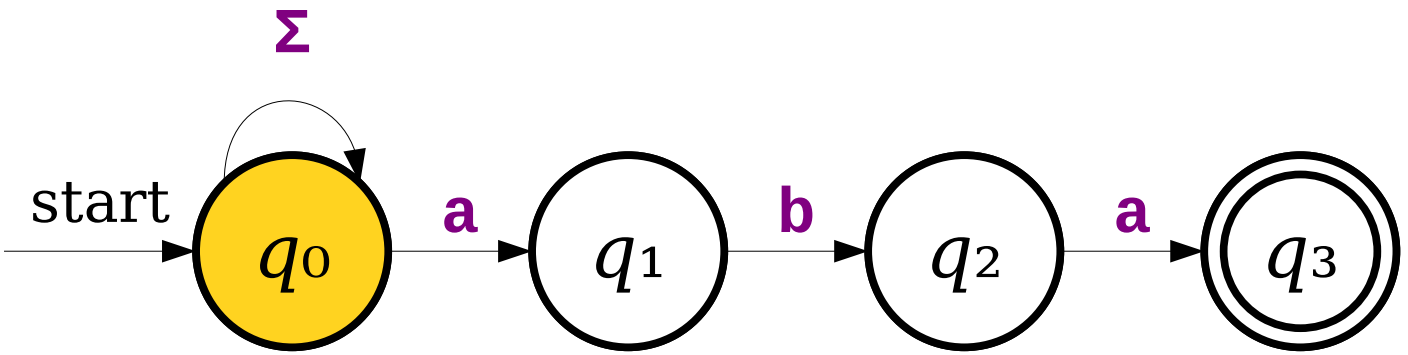

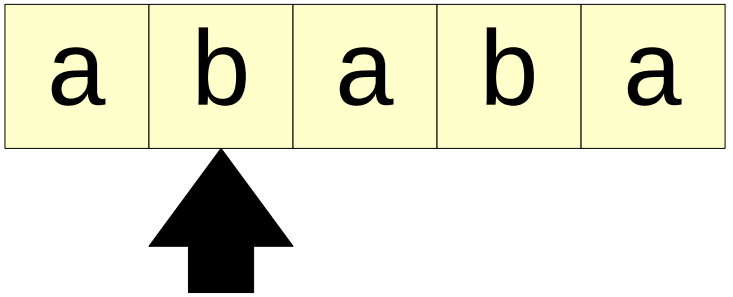

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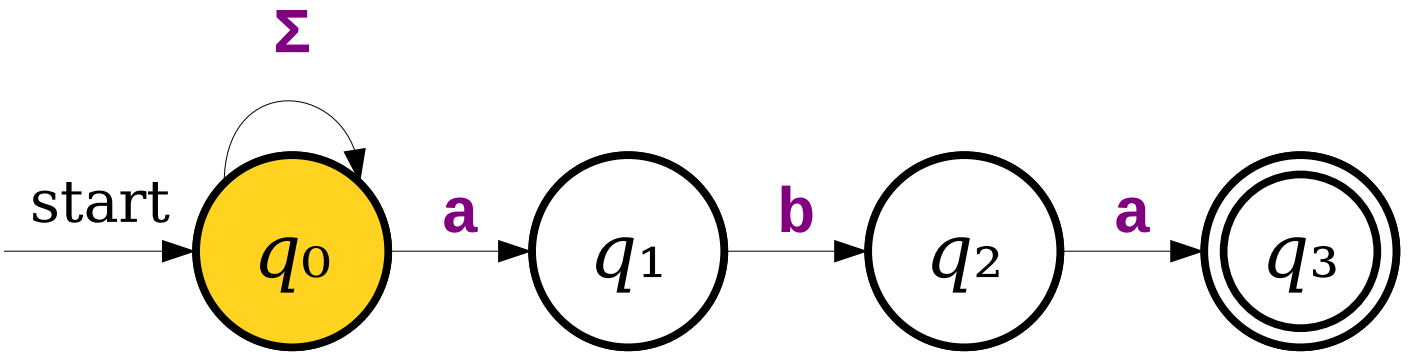

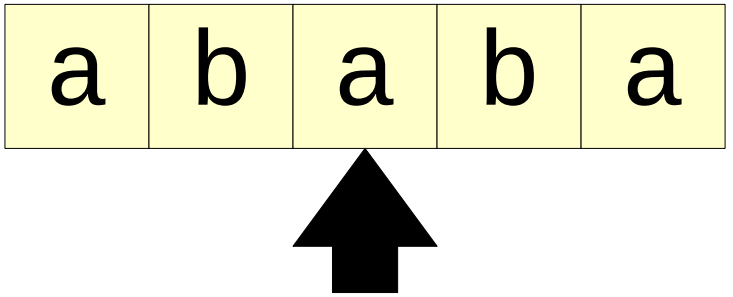

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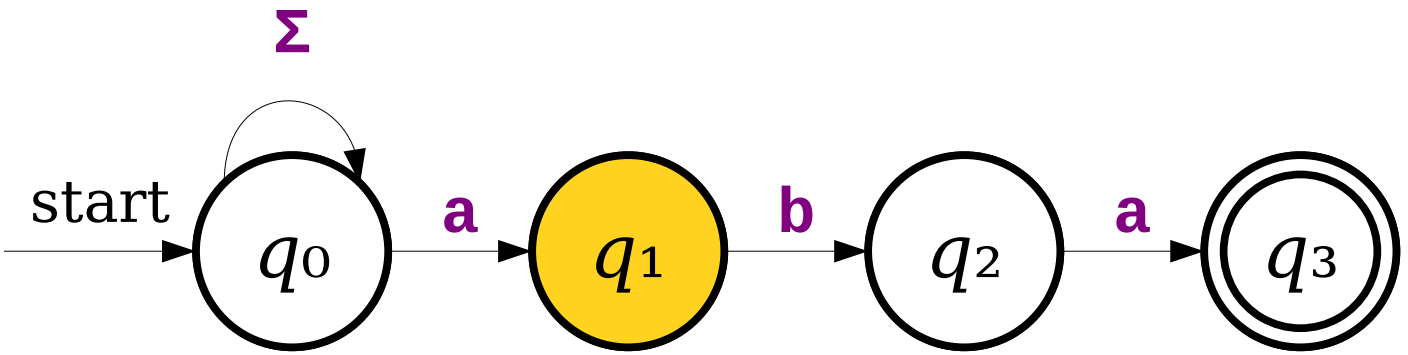

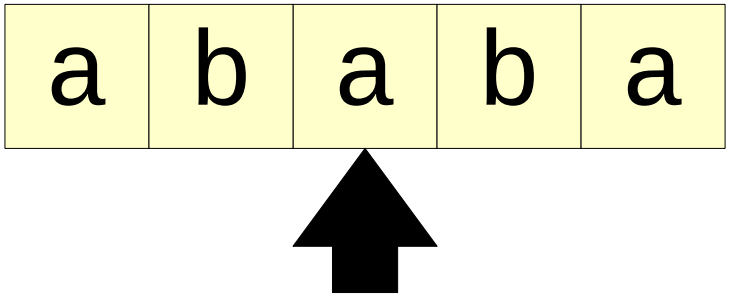

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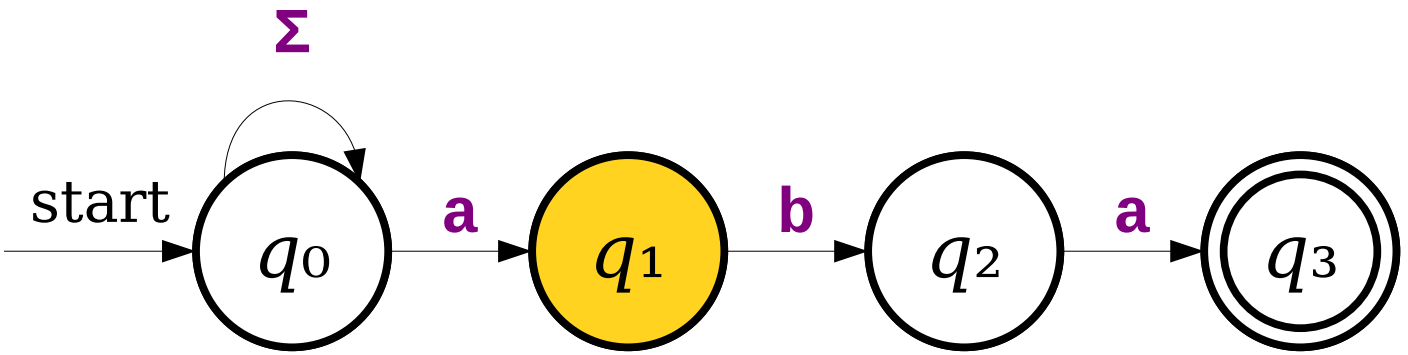

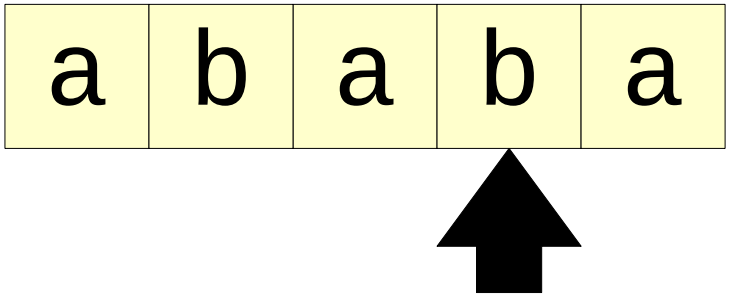

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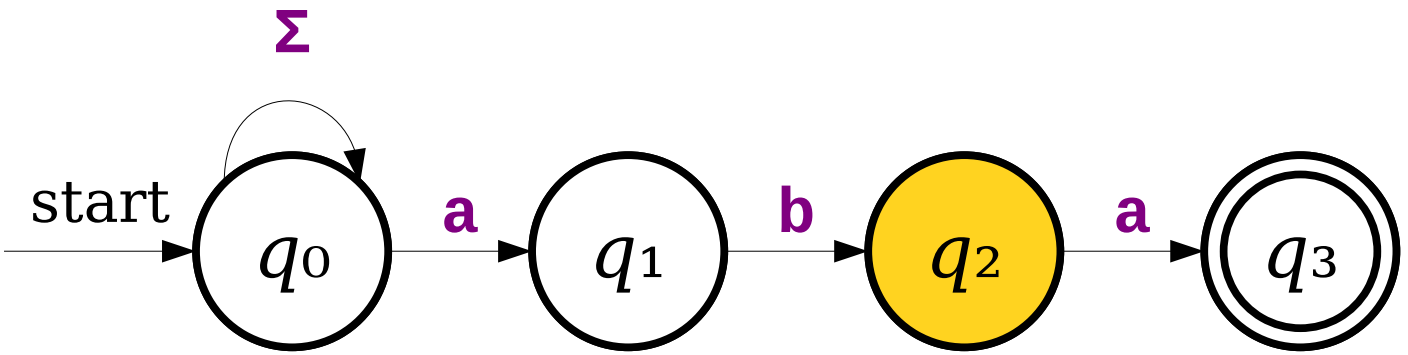

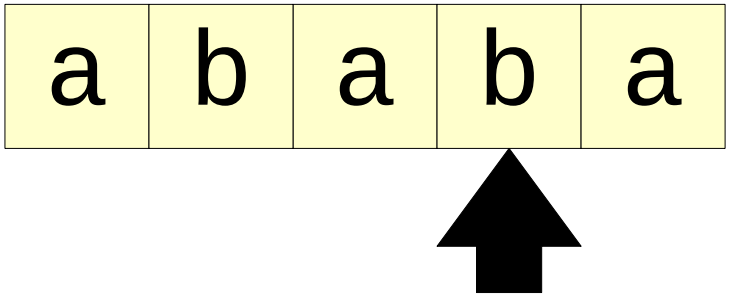

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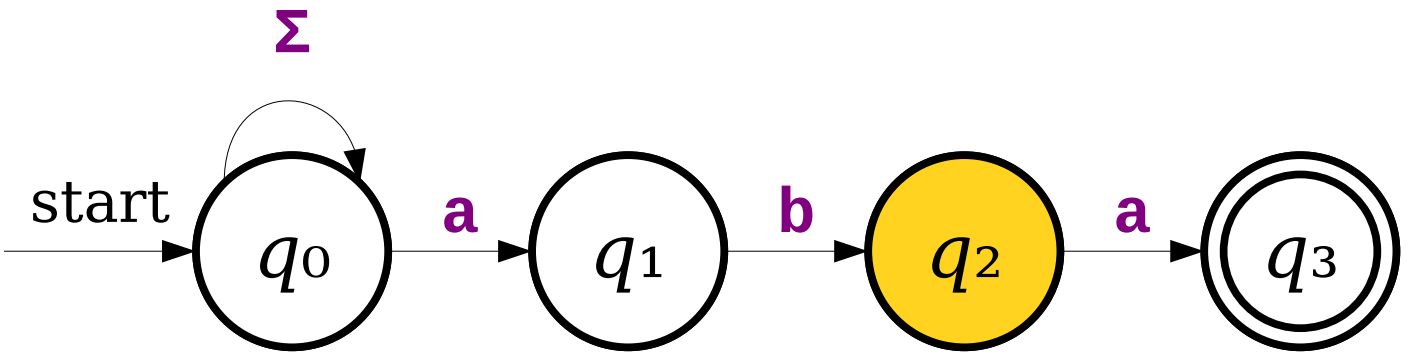

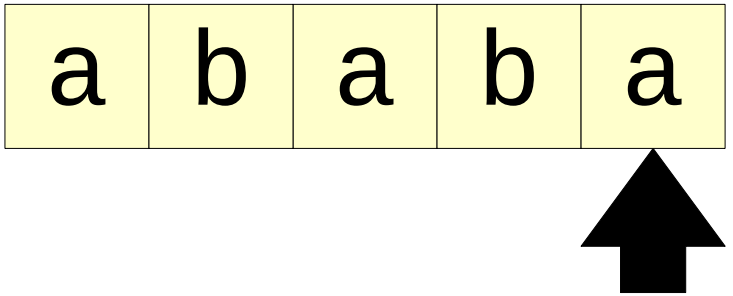

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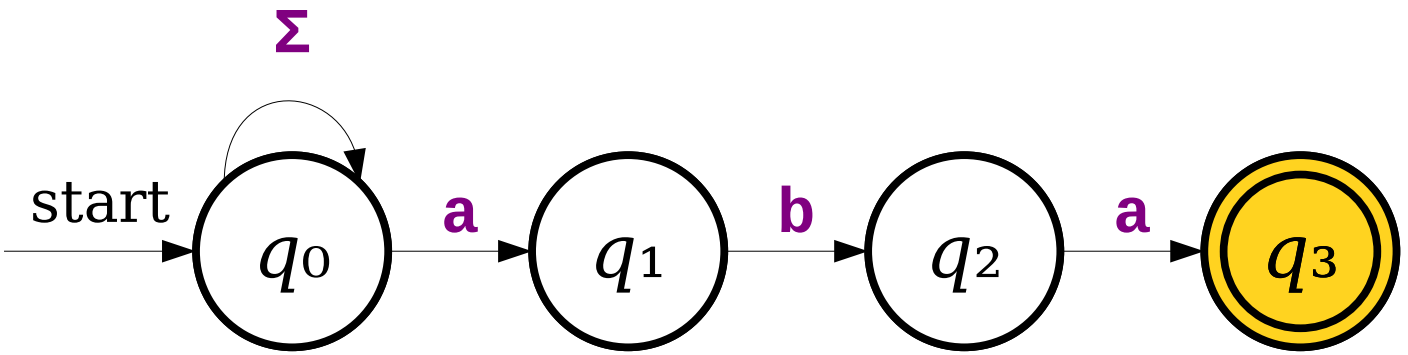

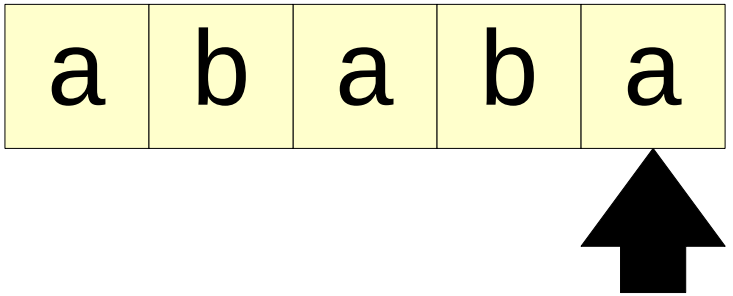

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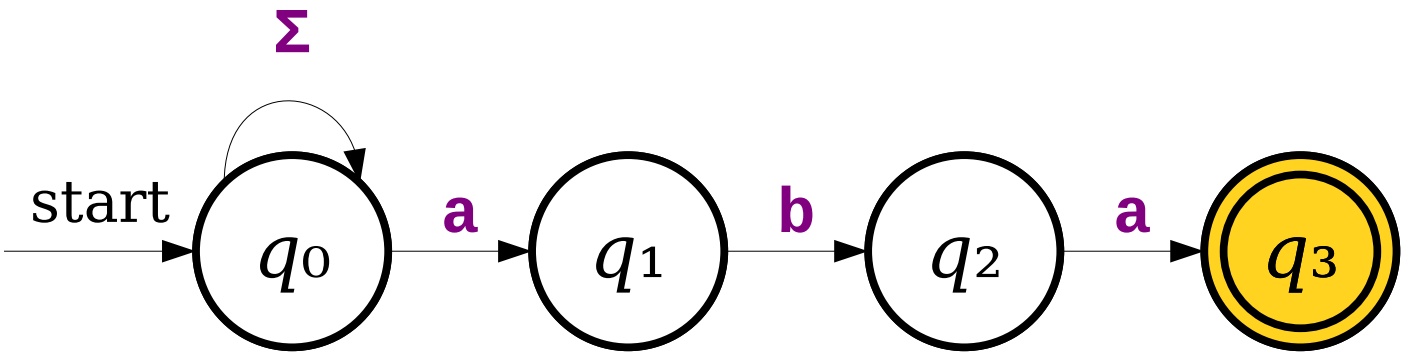

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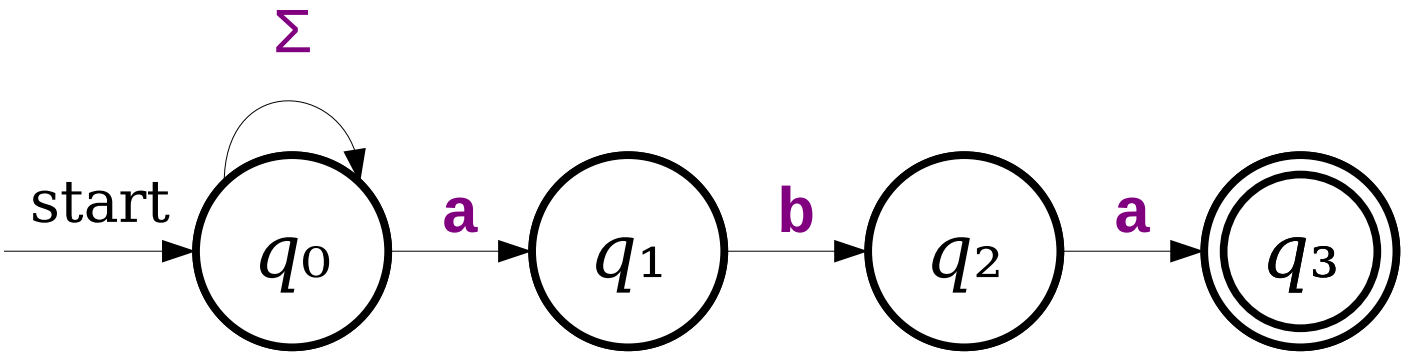
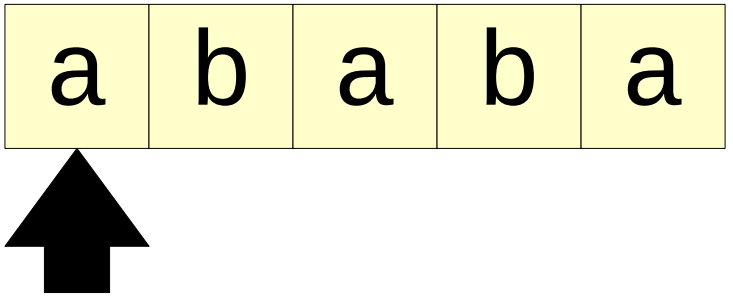
Perfect Positive Guessing

- We can view nondeterministic machines as having ***Magic Superpowers*** that enable them to guess choices that lead to an accepting state.
  - If there is at least one choice that leads to an accepting state, the machine will guess it.
  - If there are no choices, the machine guesses any one of the wrong guesses.
- There is no known way to physically model this intuition of nondeterminism – this is quite a departure from reality!

Massive Parallelism





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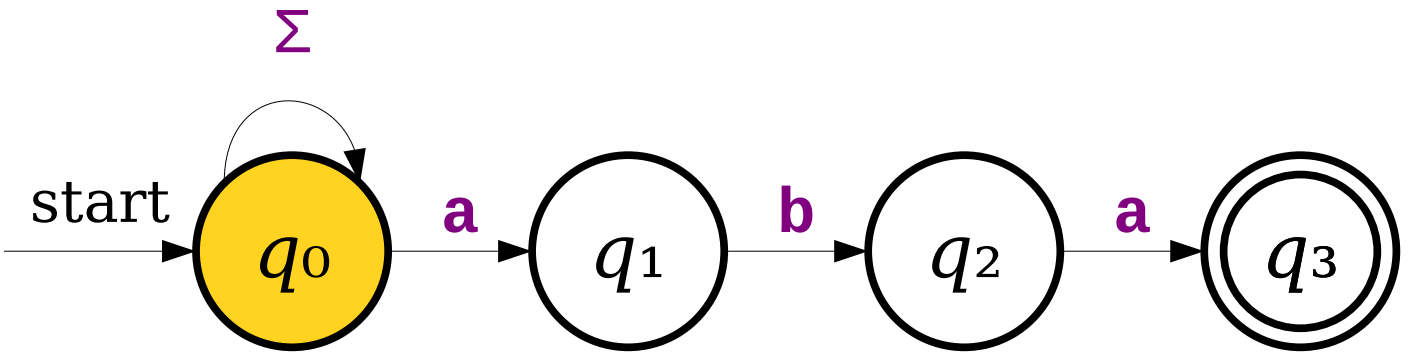

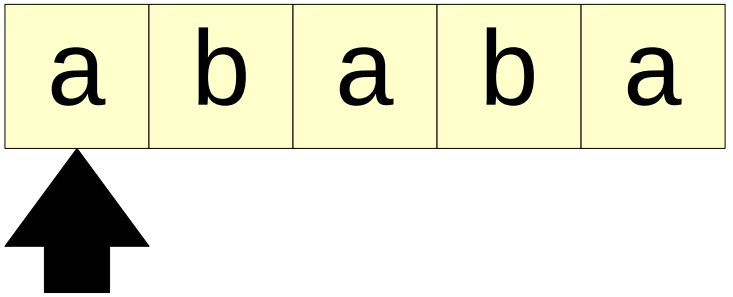

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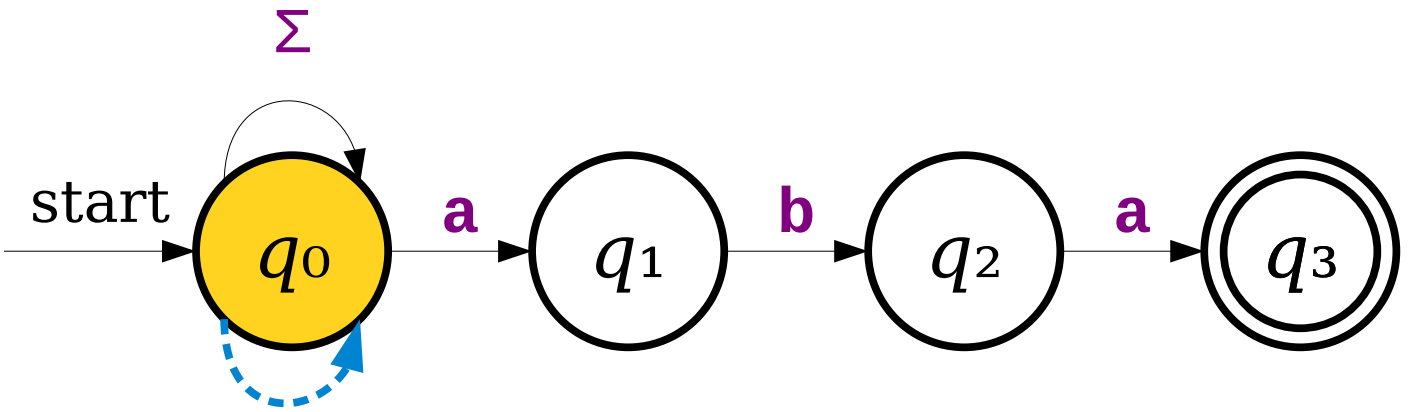

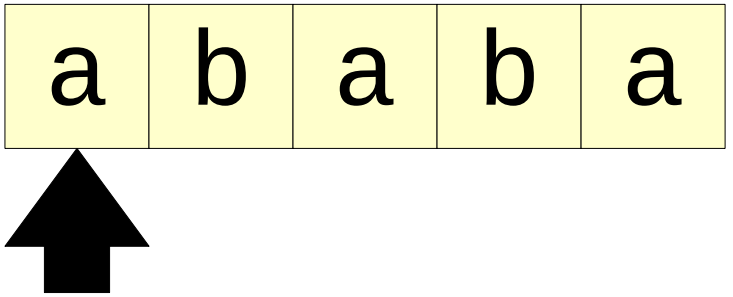

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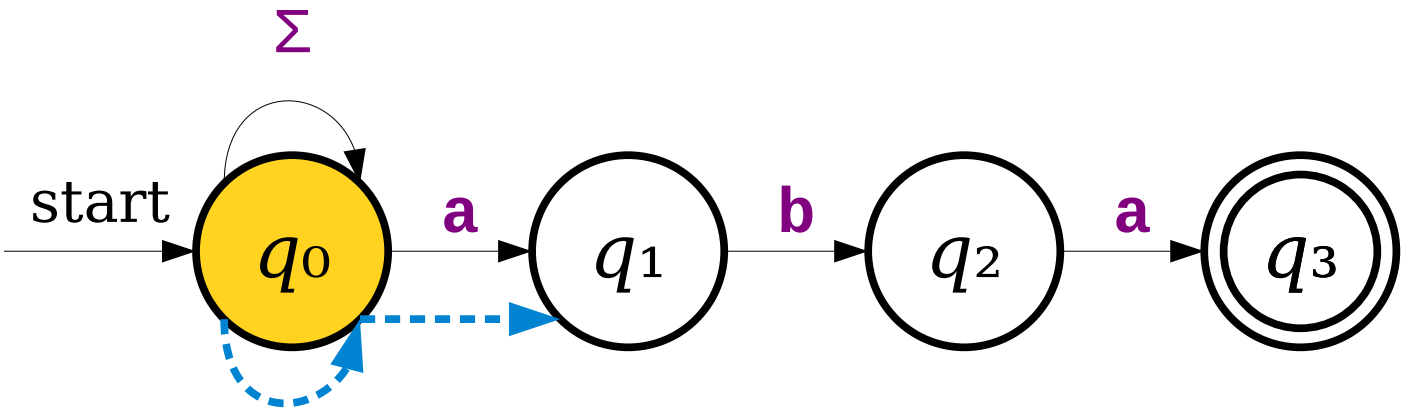

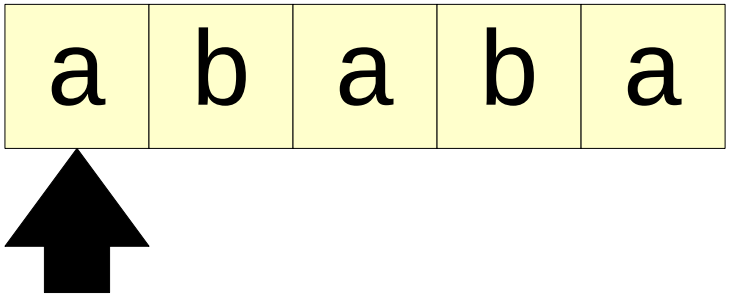

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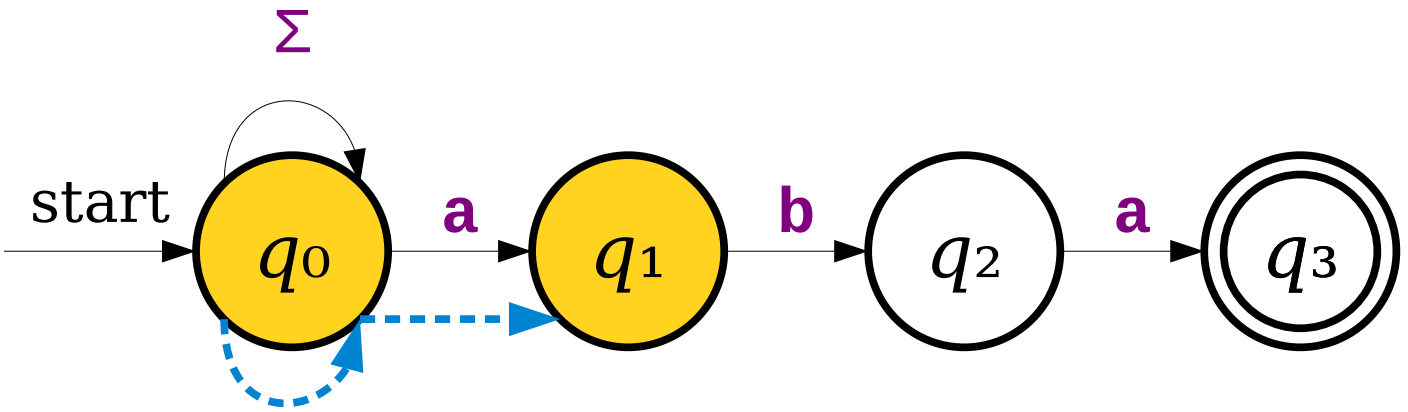

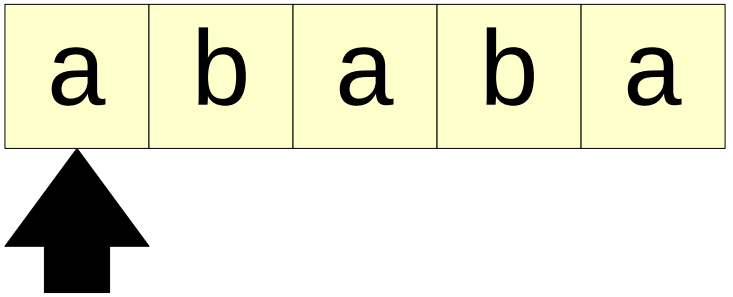

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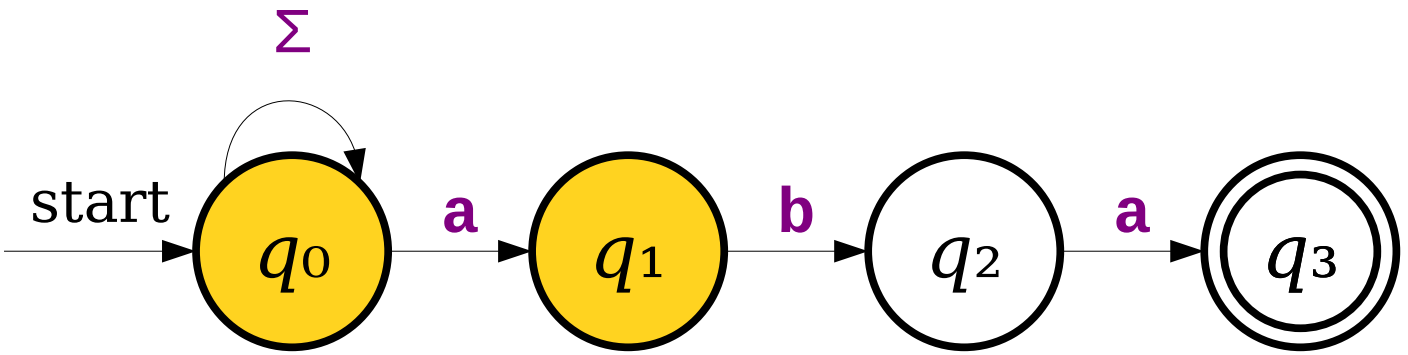

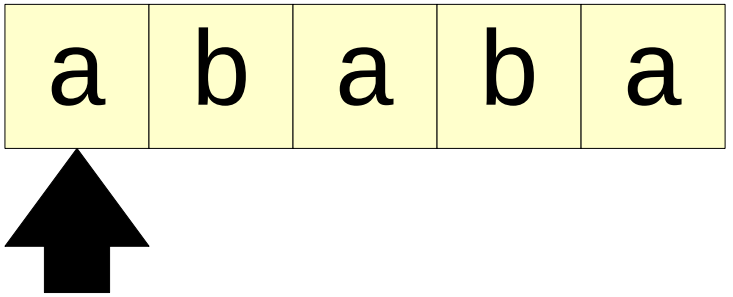

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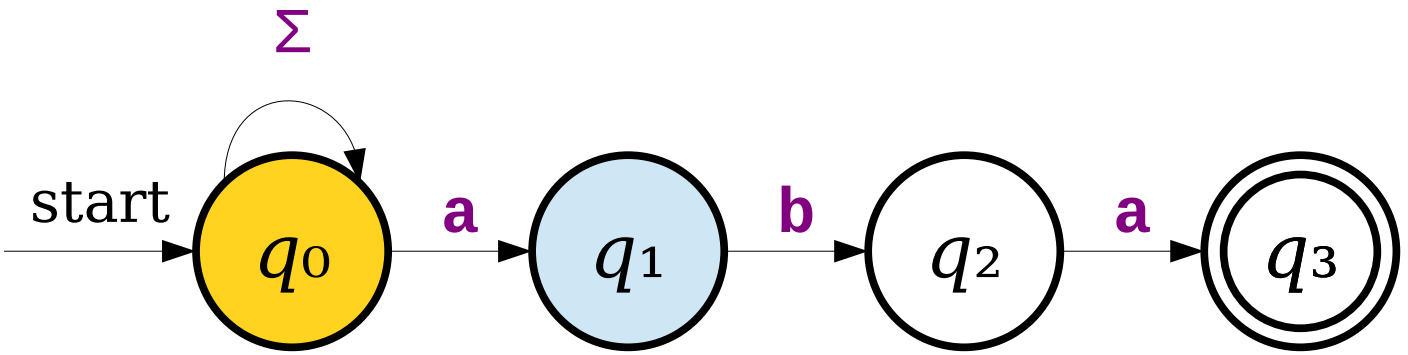

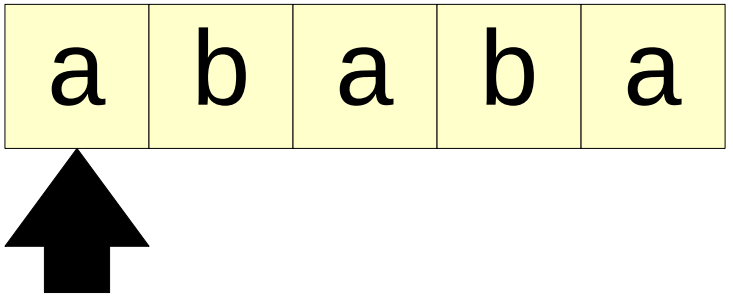

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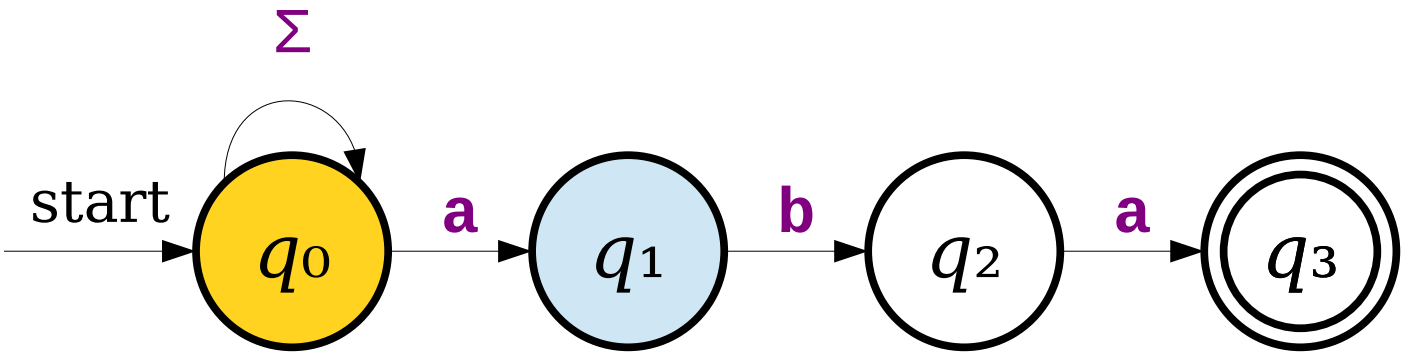

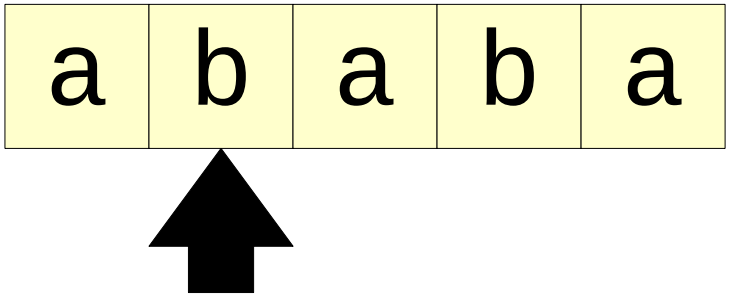

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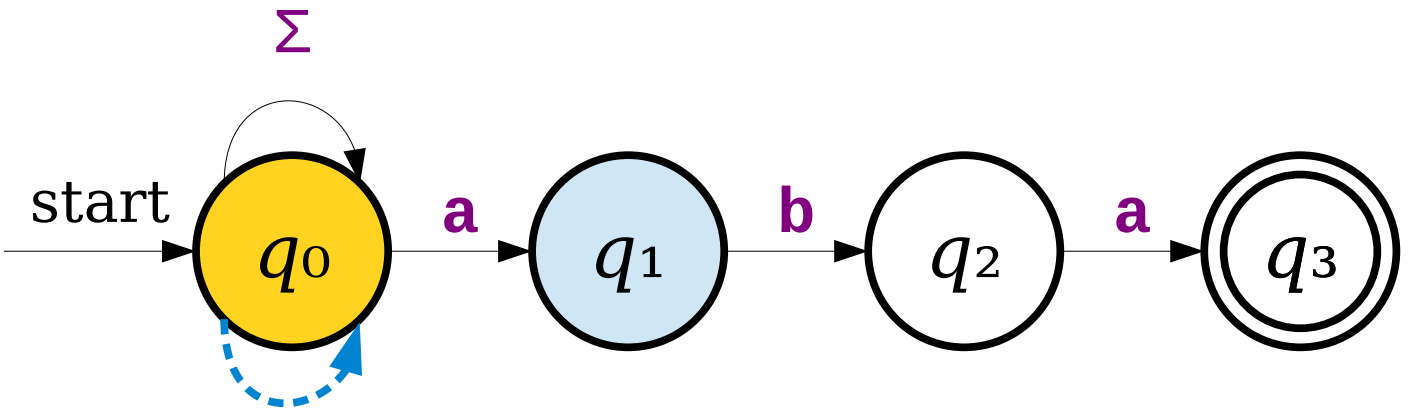

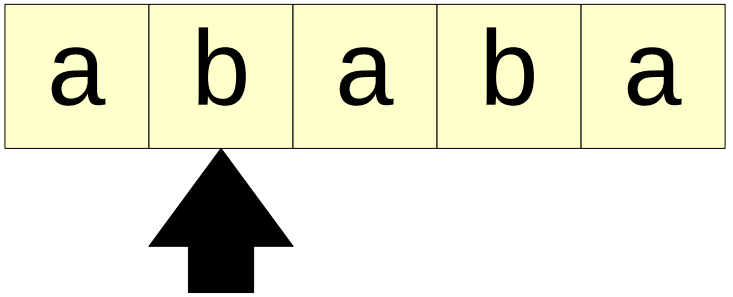

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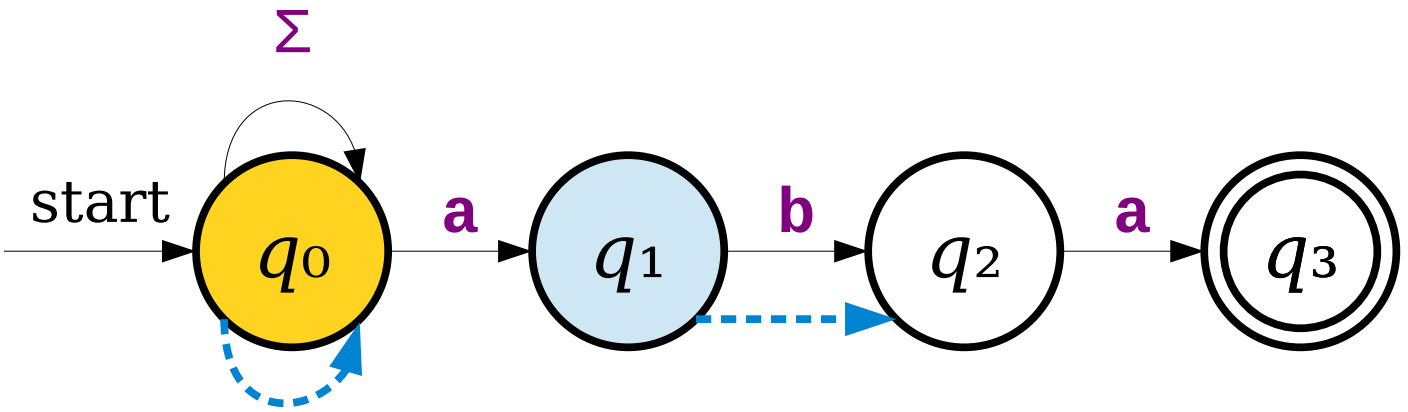

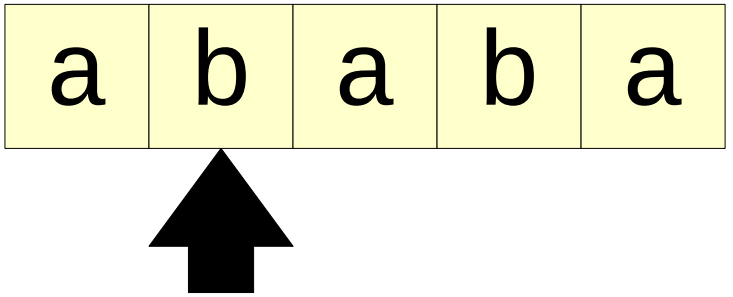

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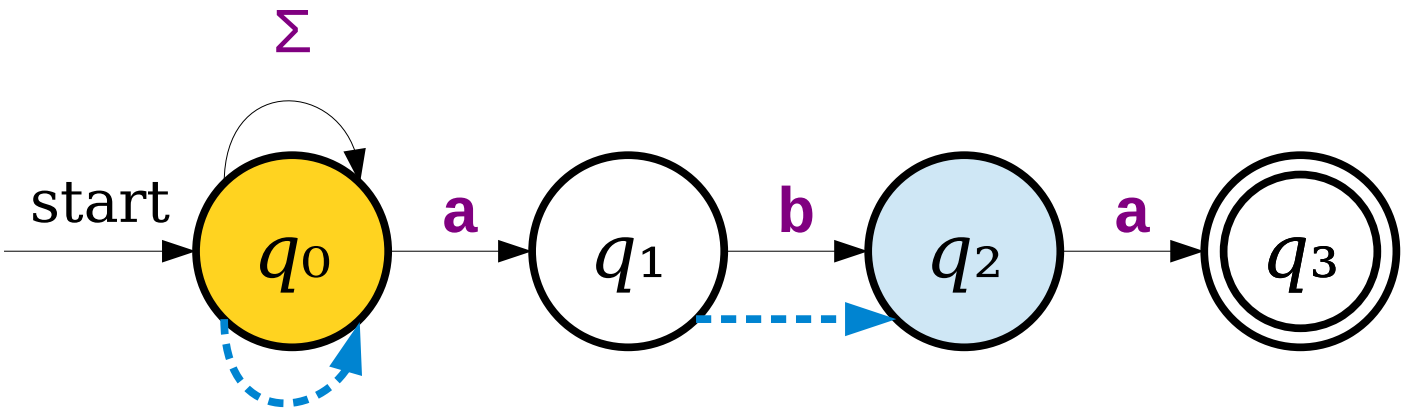

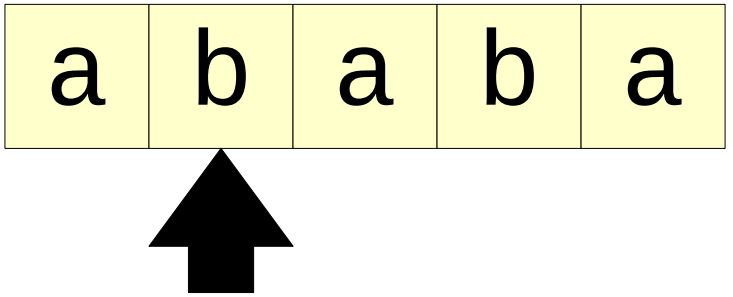

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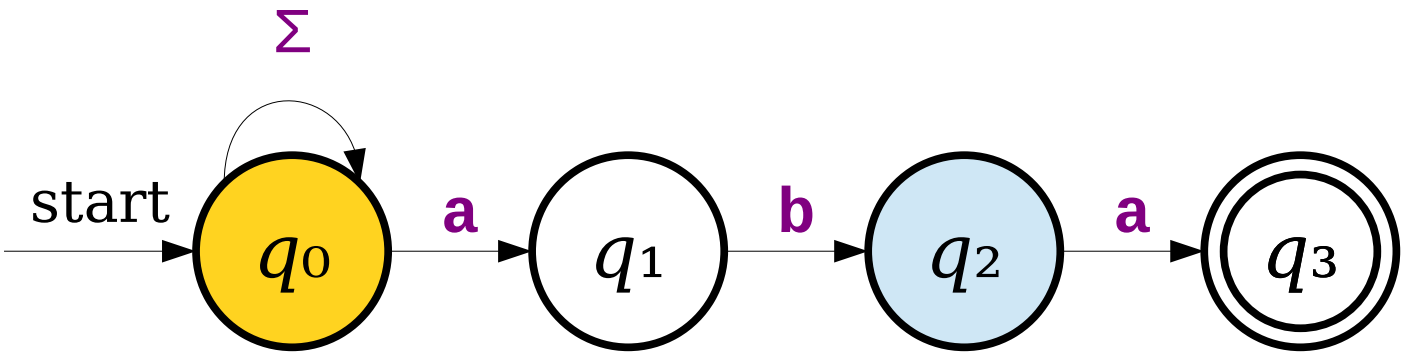

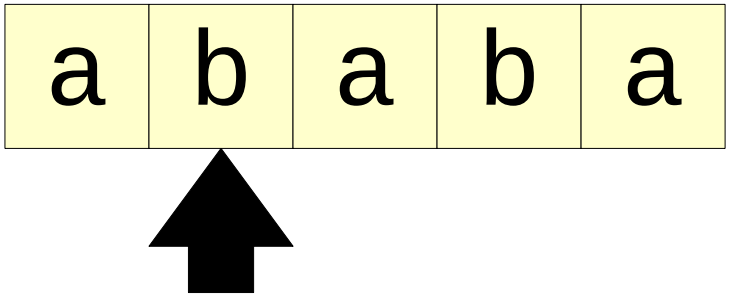

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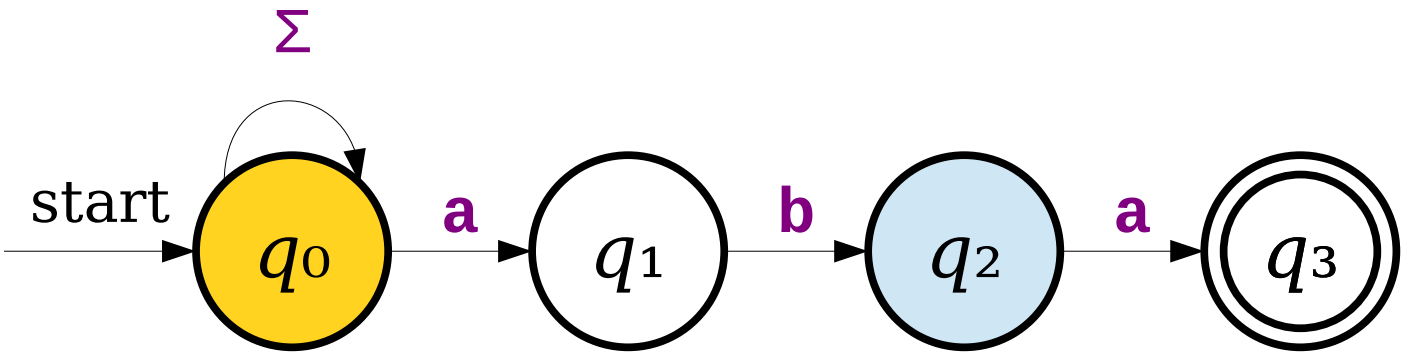

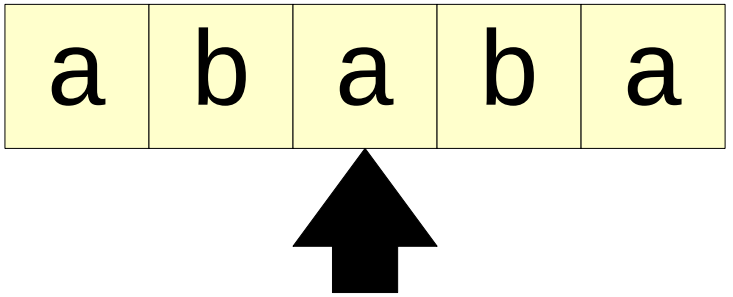

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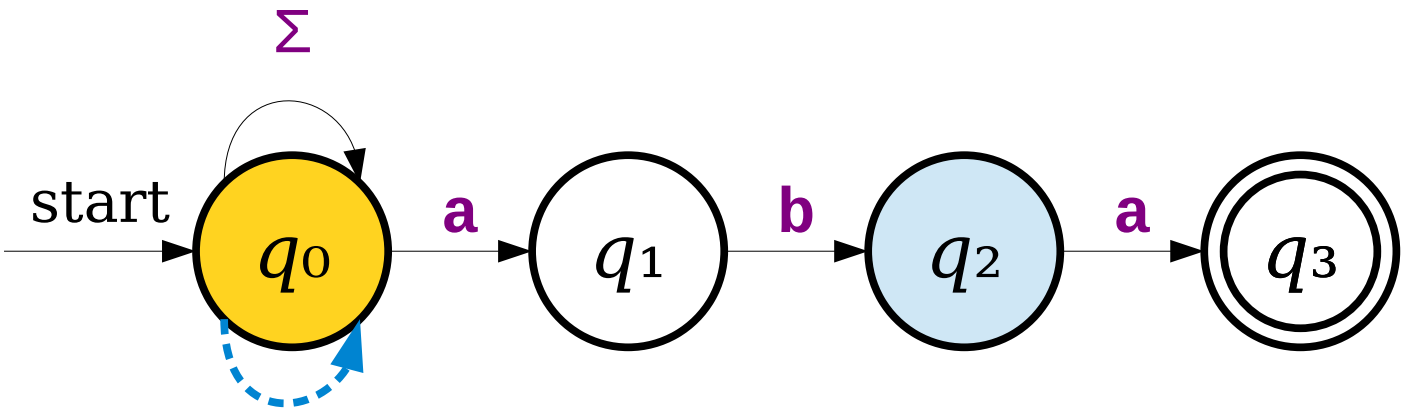

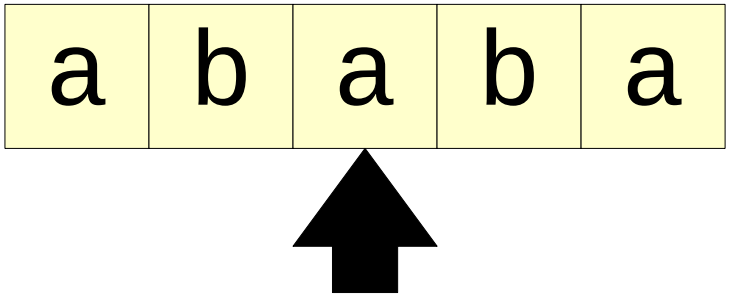

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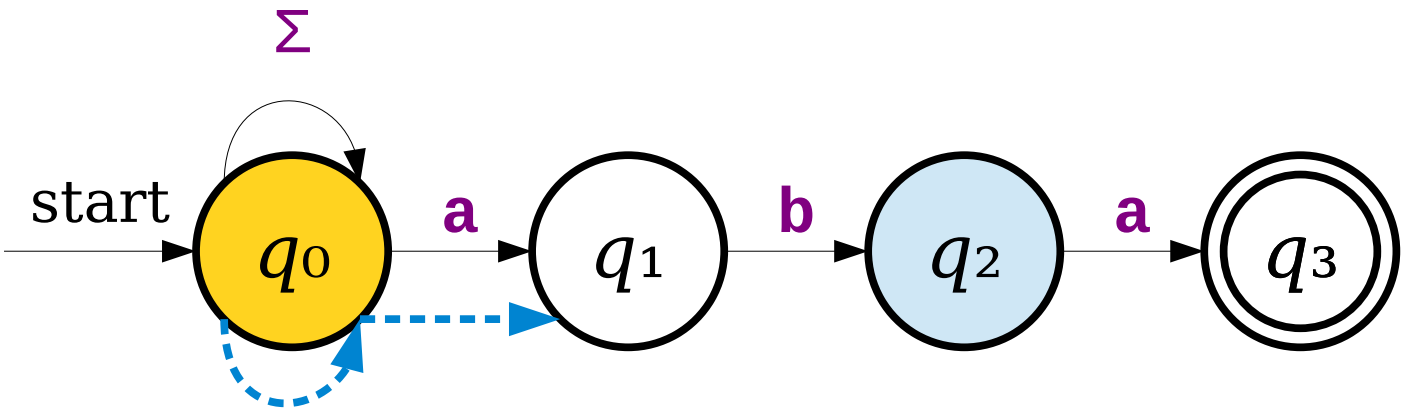

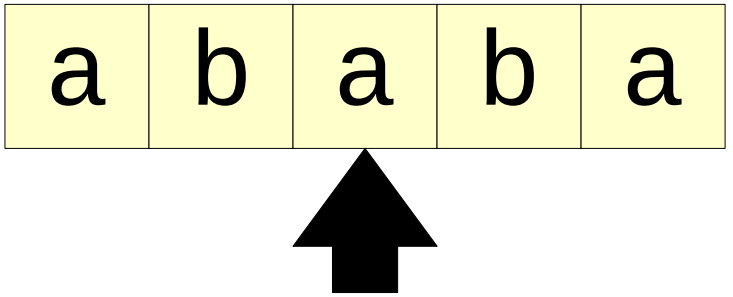

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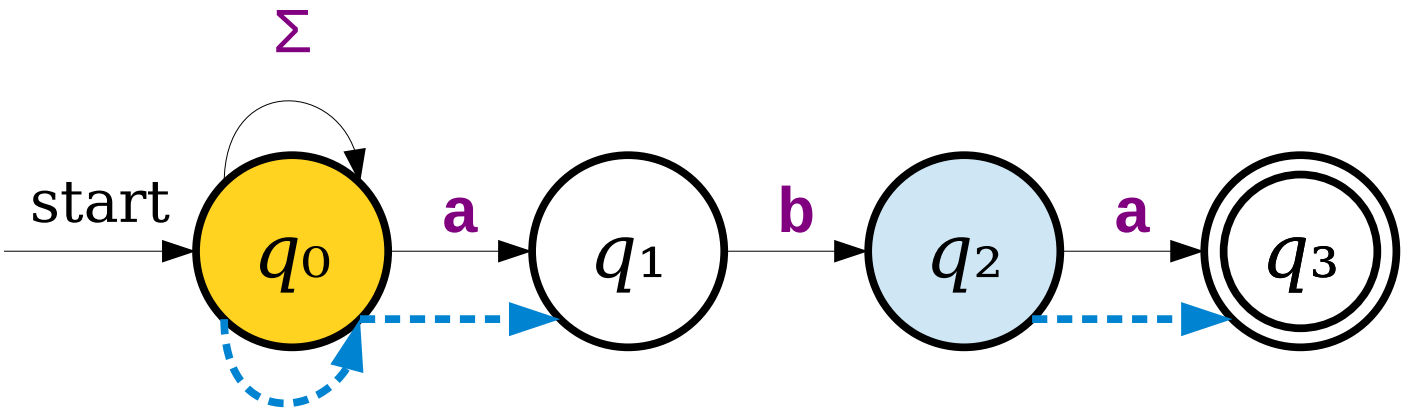

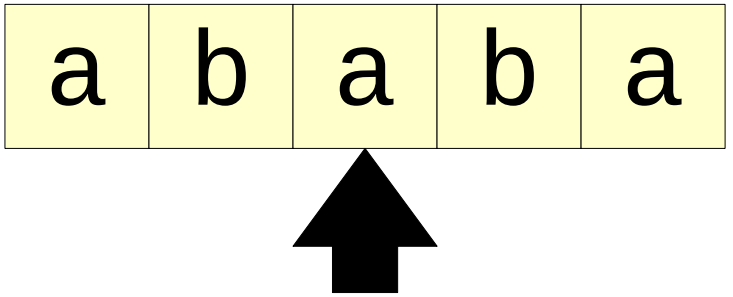

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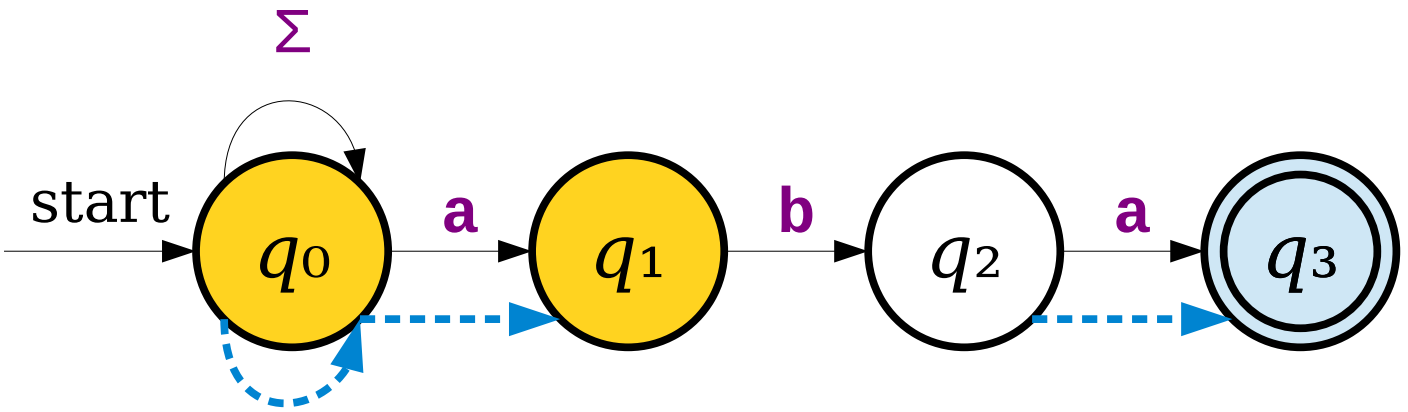

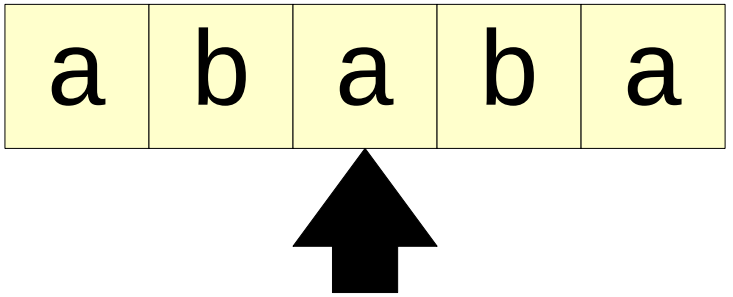

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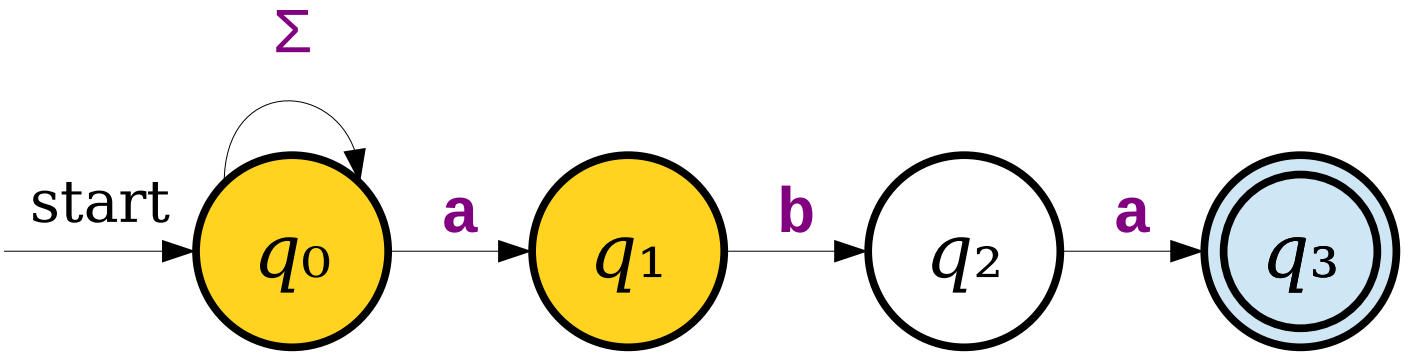

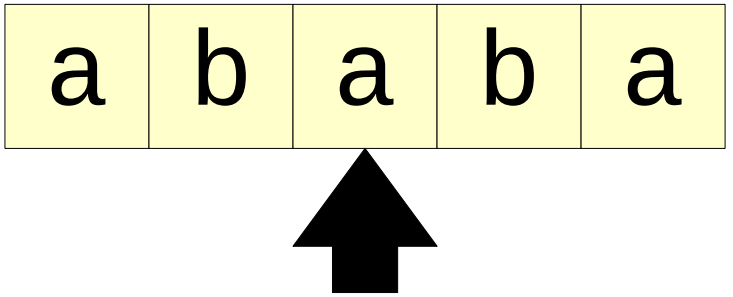

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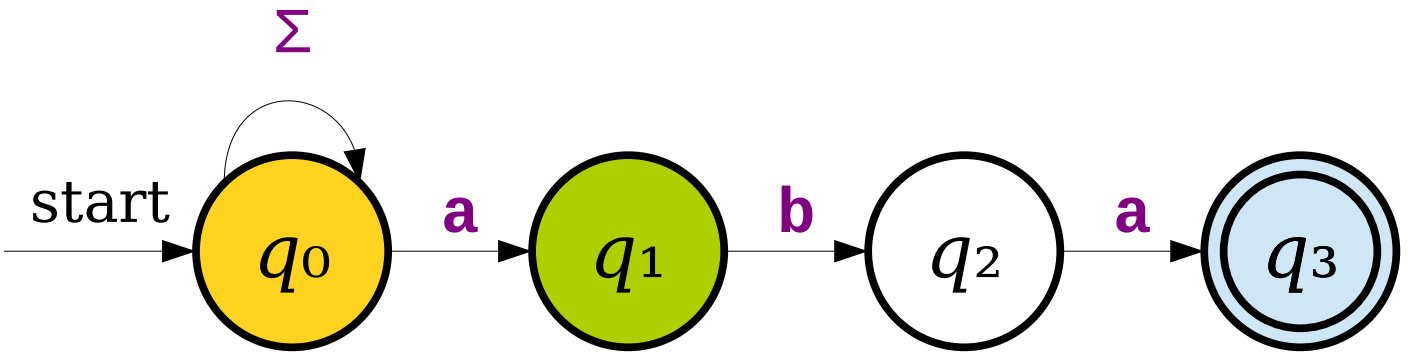

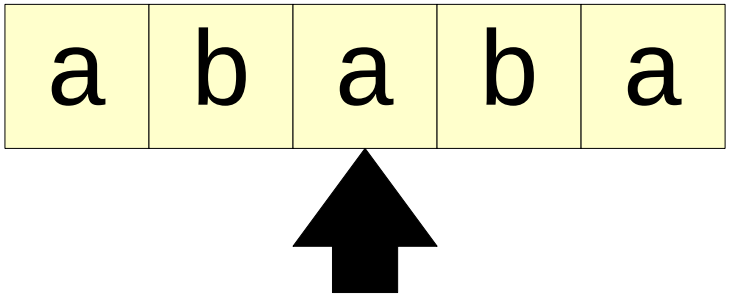

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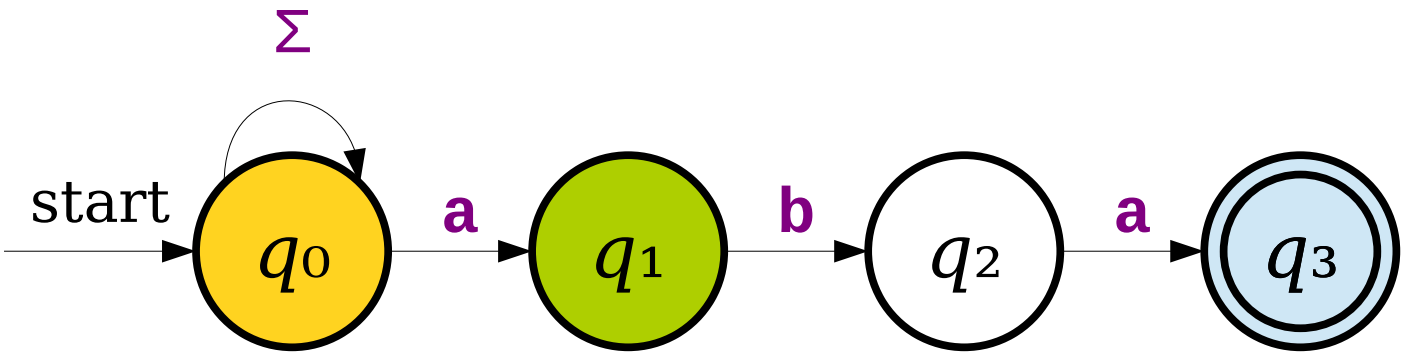

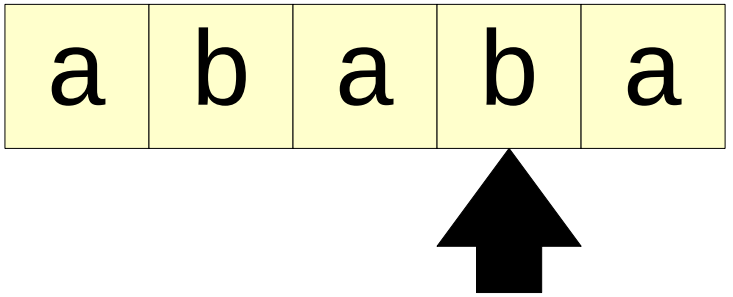

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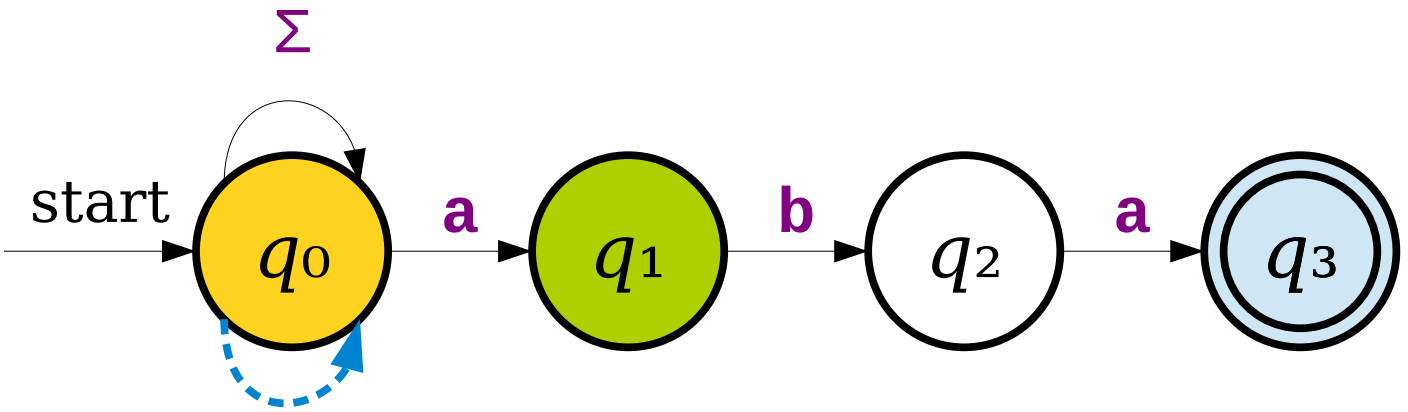

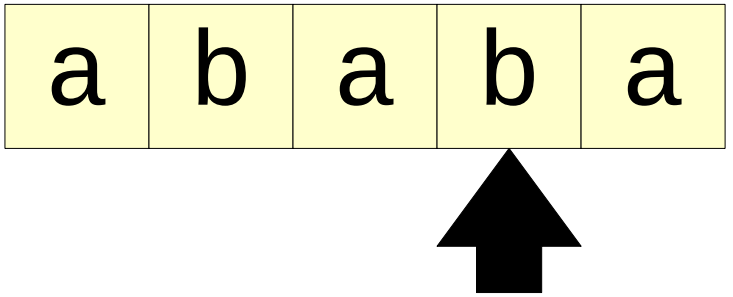

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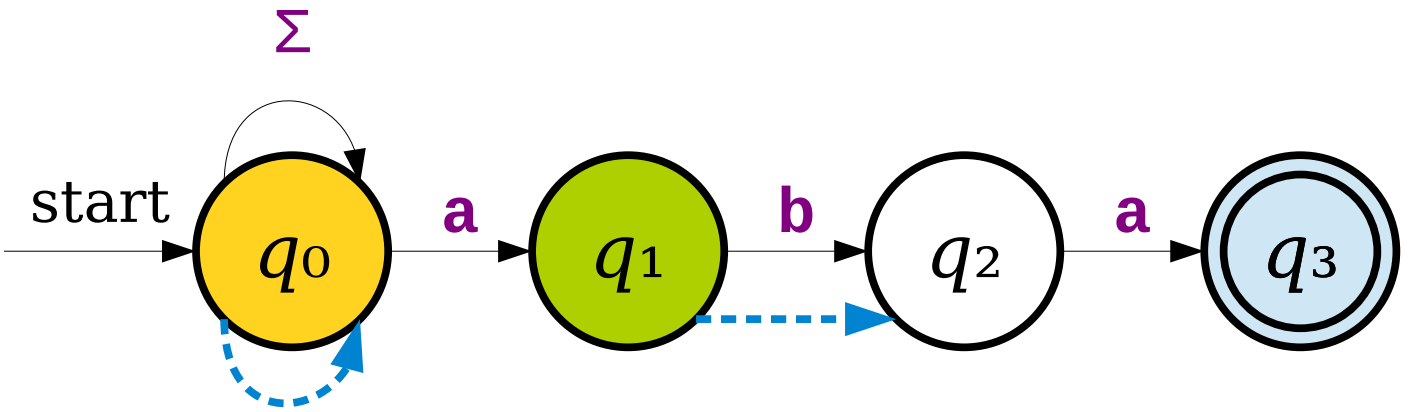

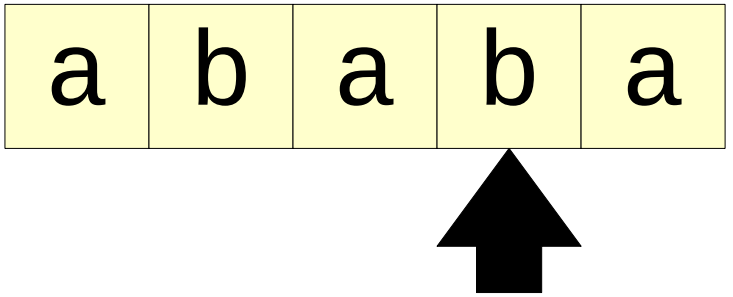

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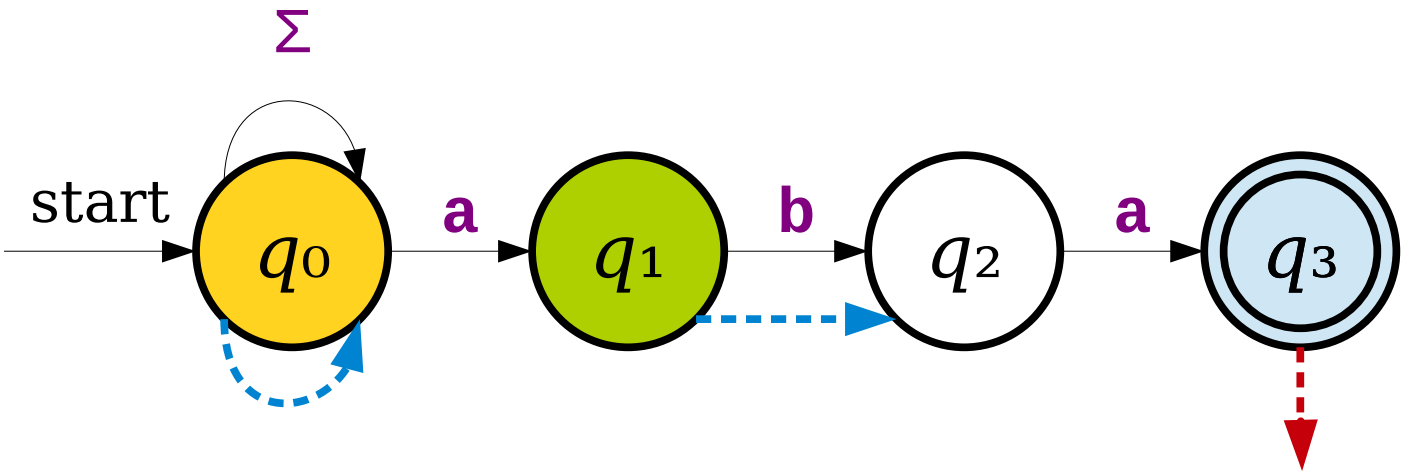

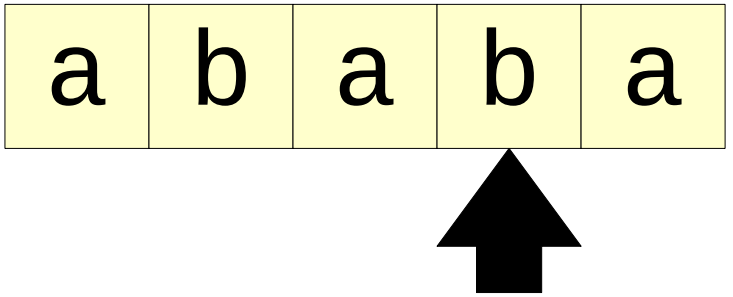

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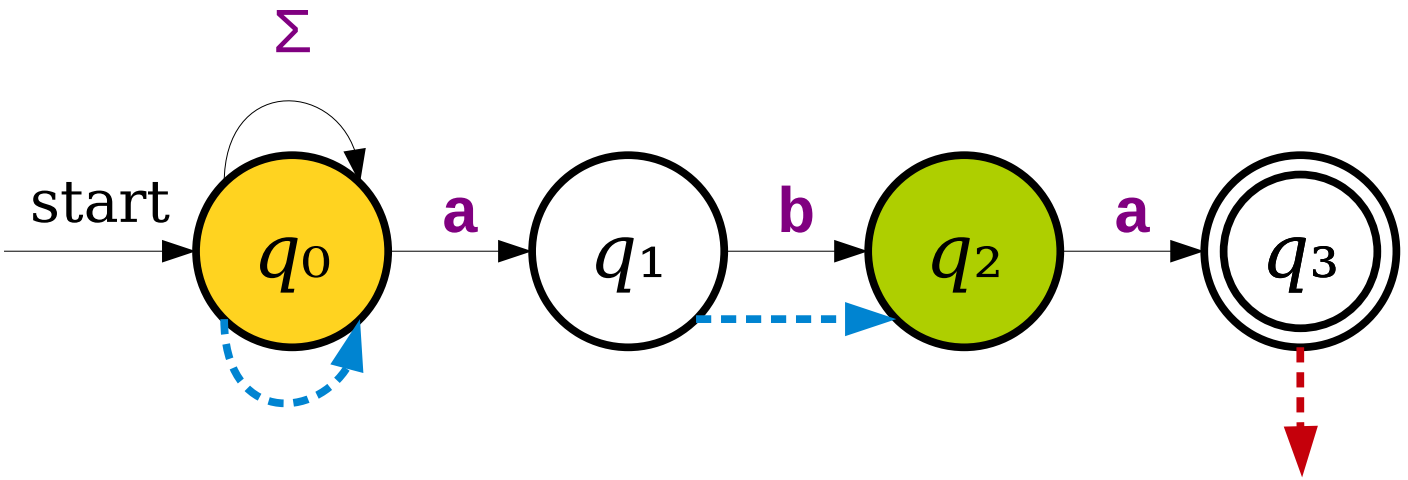

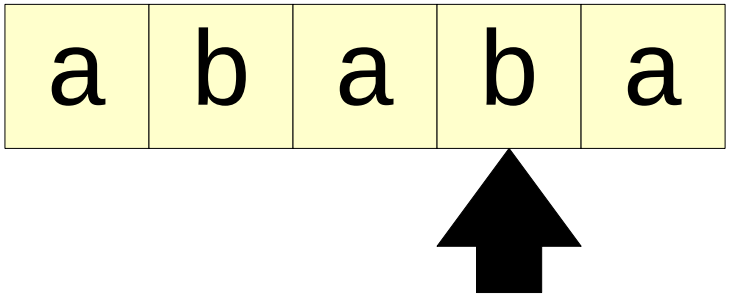

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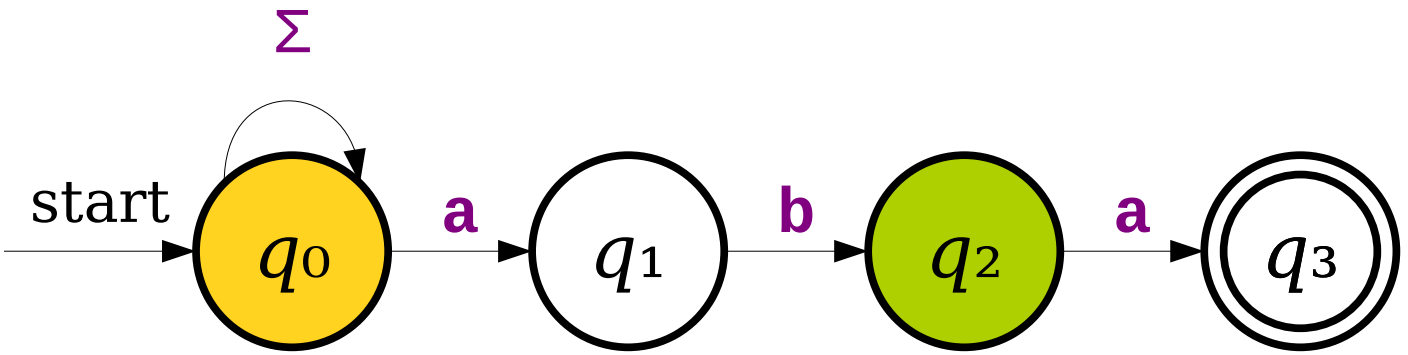
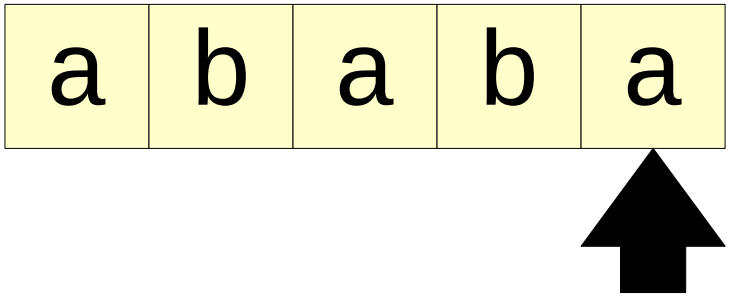
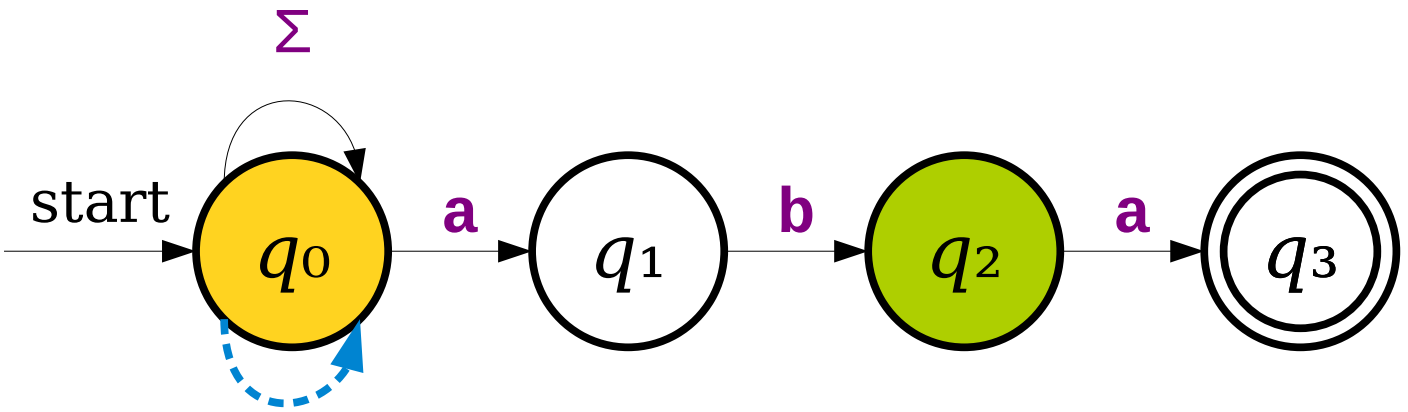


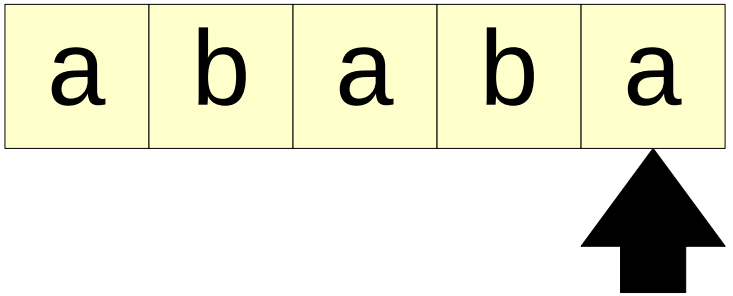


Using the massive parallelism intuition, if we are in the states q_0 and q_2, what set of states will we be in after reading the character **a**?

Respond at pollev.com/zhenglian740

Massive Parallelism





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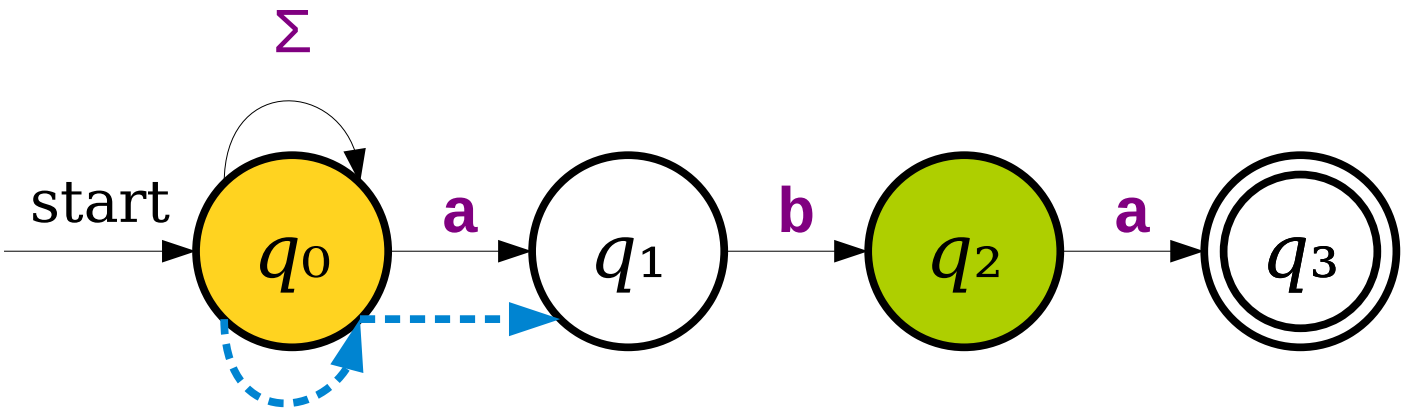

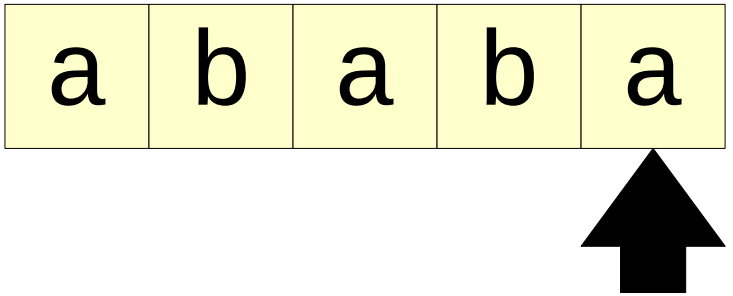

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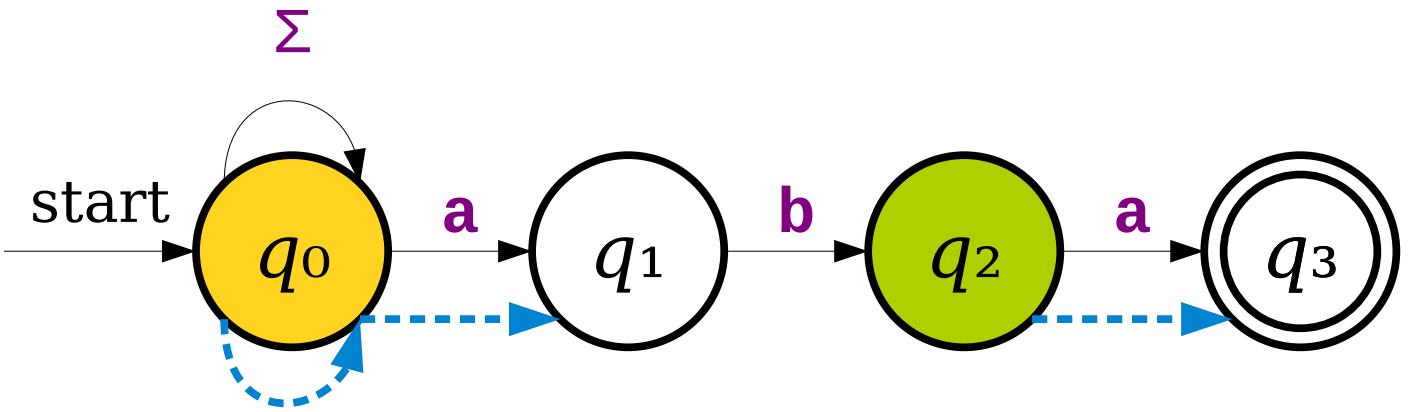

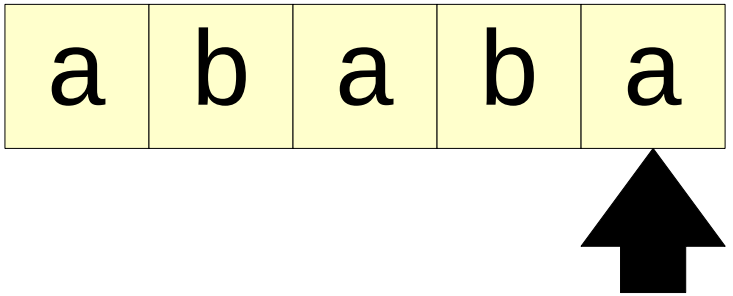

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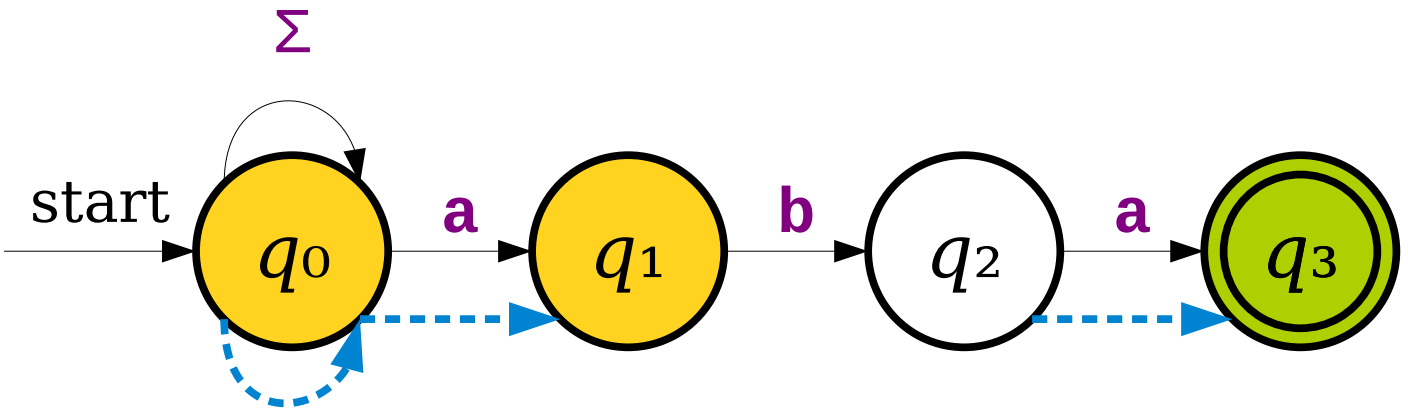

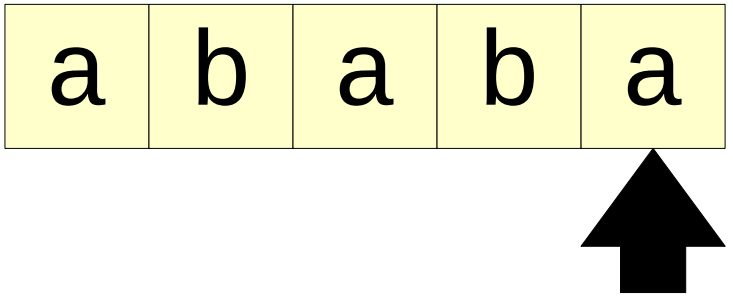

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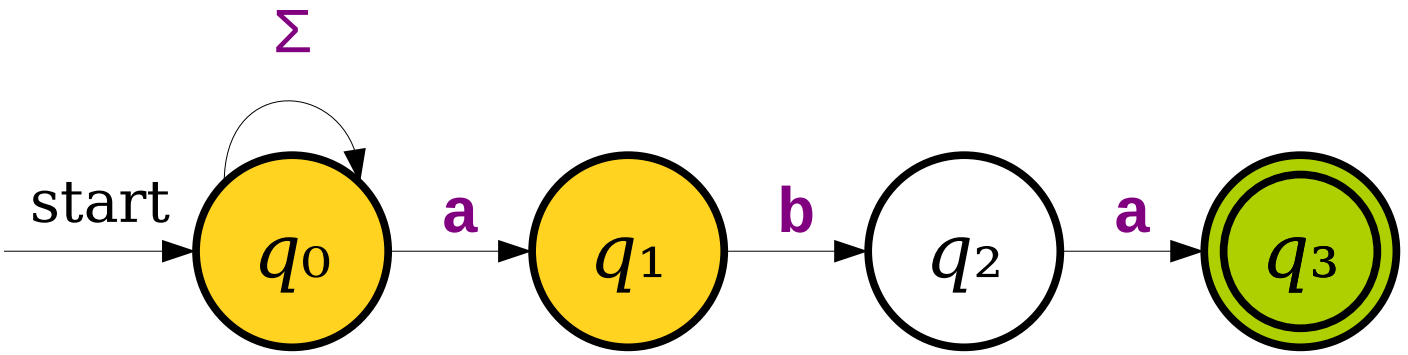
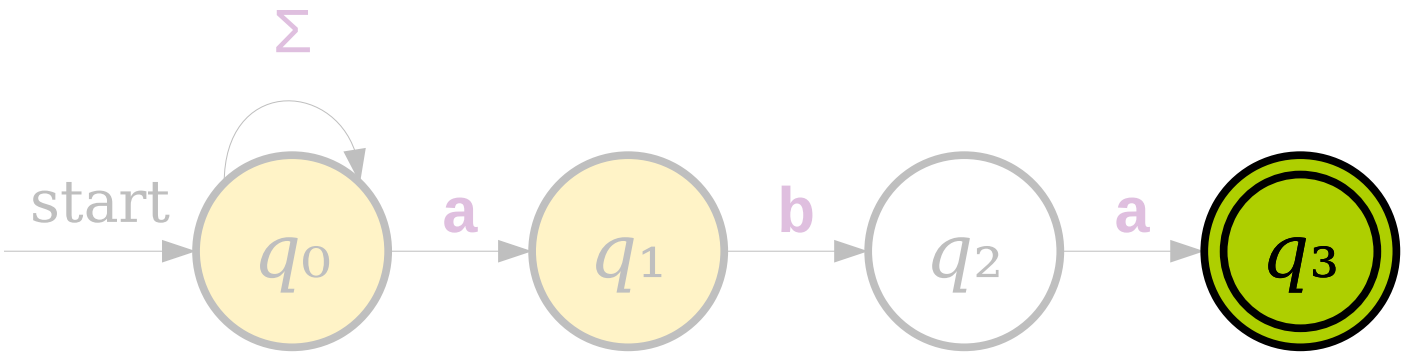


a	b	a	b	a

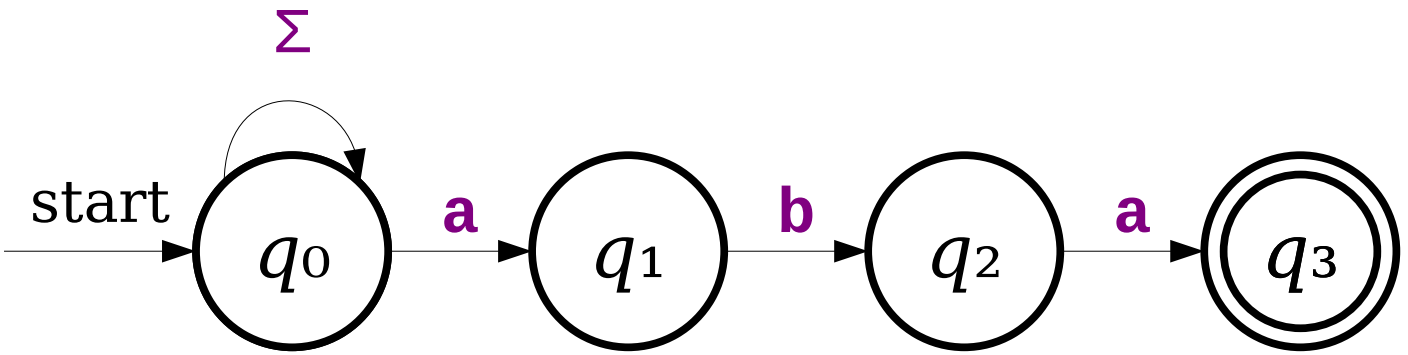
Massive Parallelism

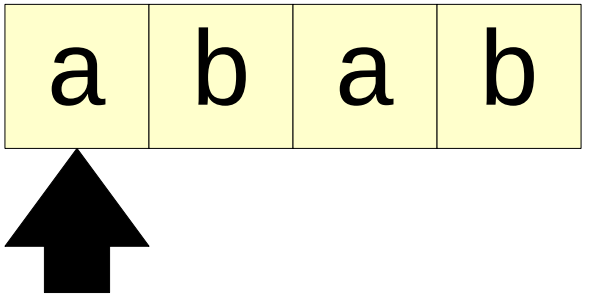


We're in at least one accepting state, so there's some path that gets us to an accepting state.

a	b	a	b	a

Massive Parallelism





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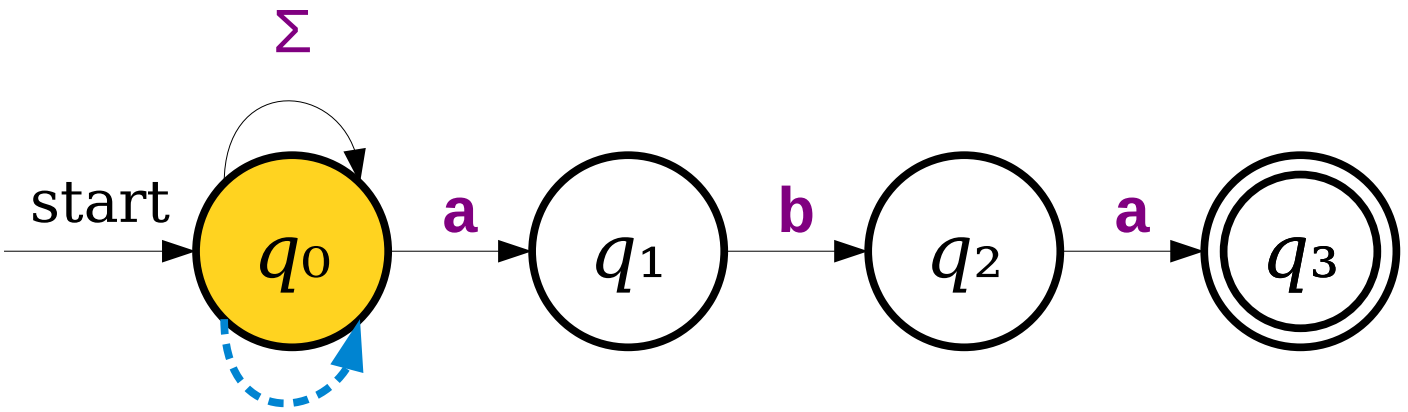

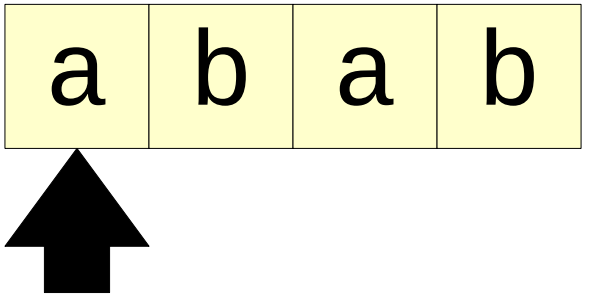

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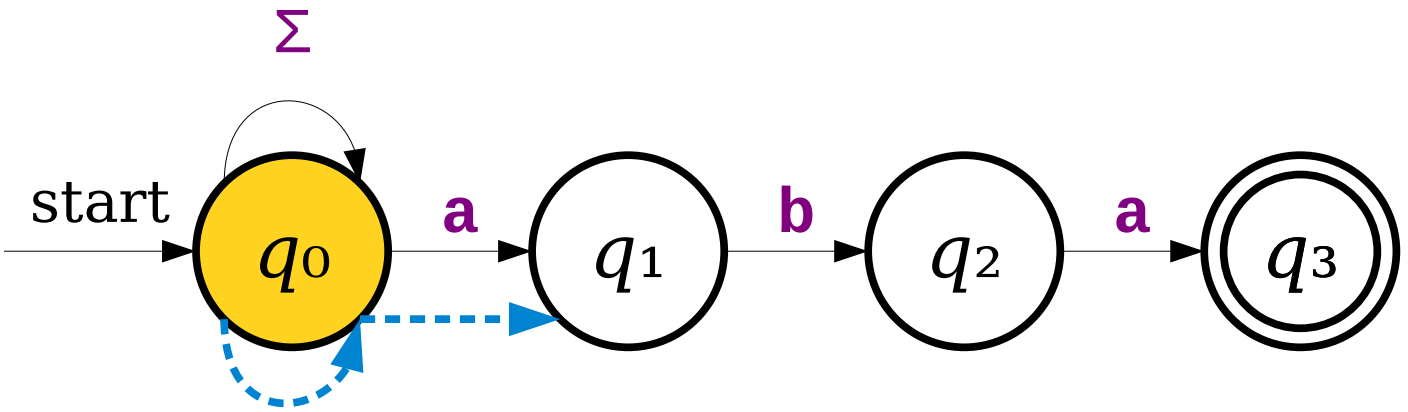

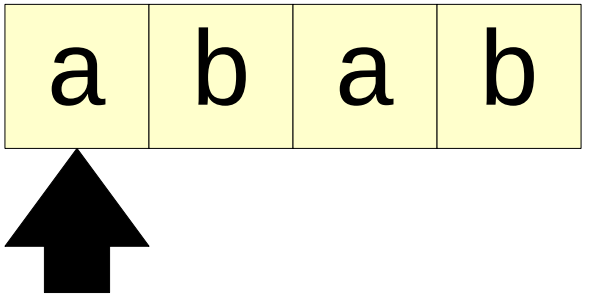

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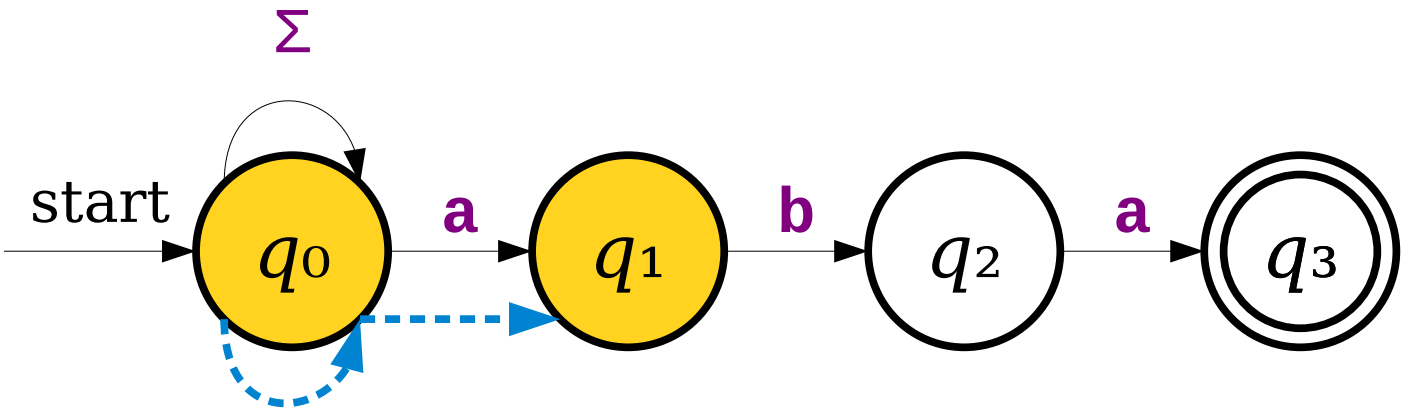

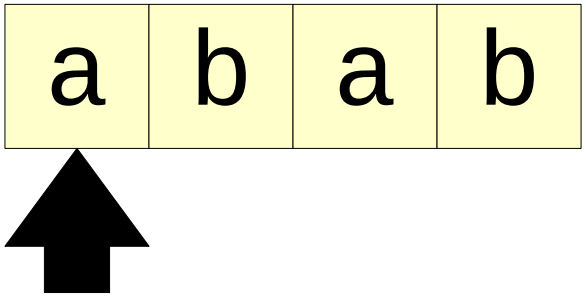

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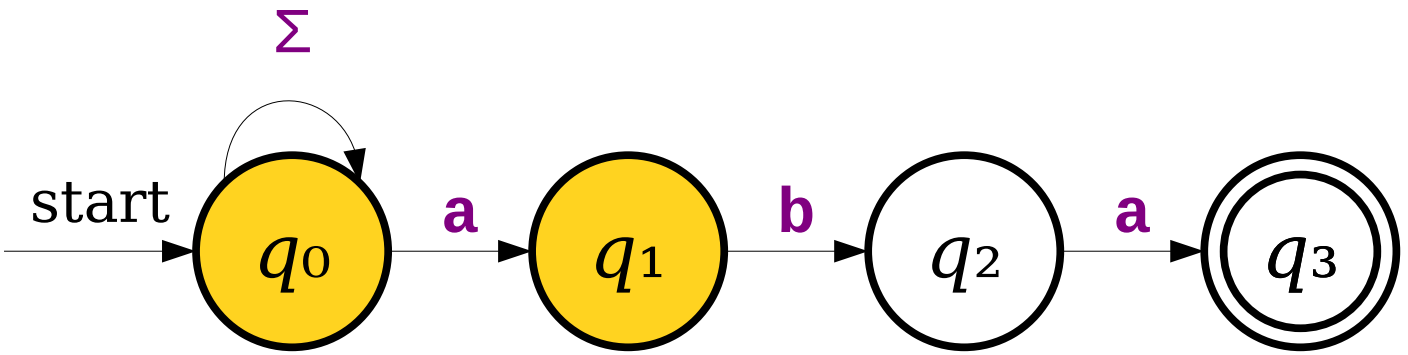

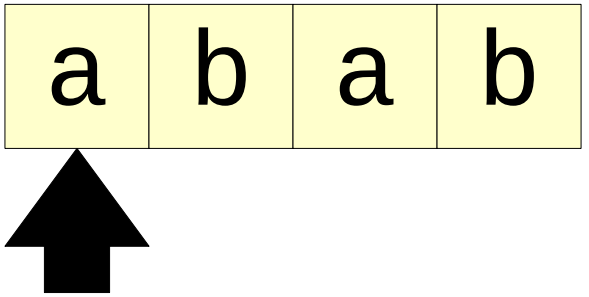

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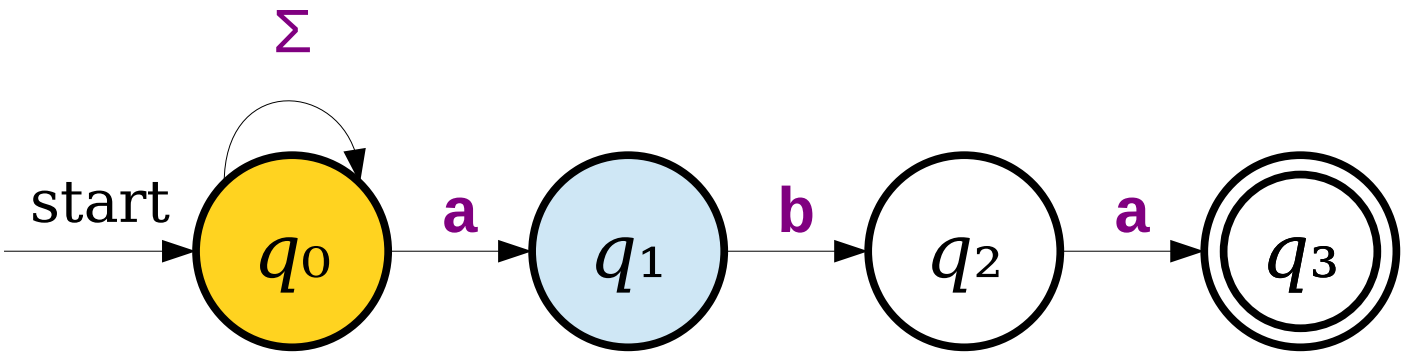

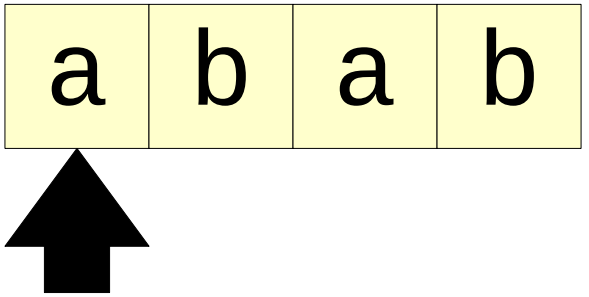

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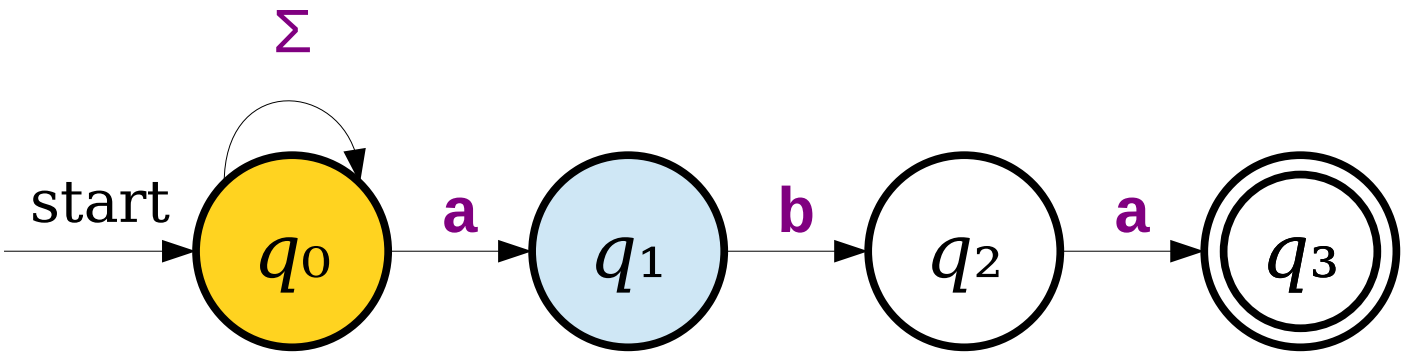

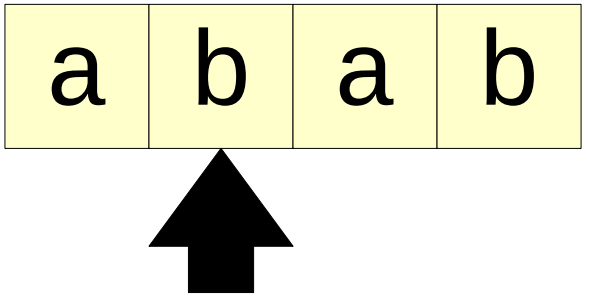

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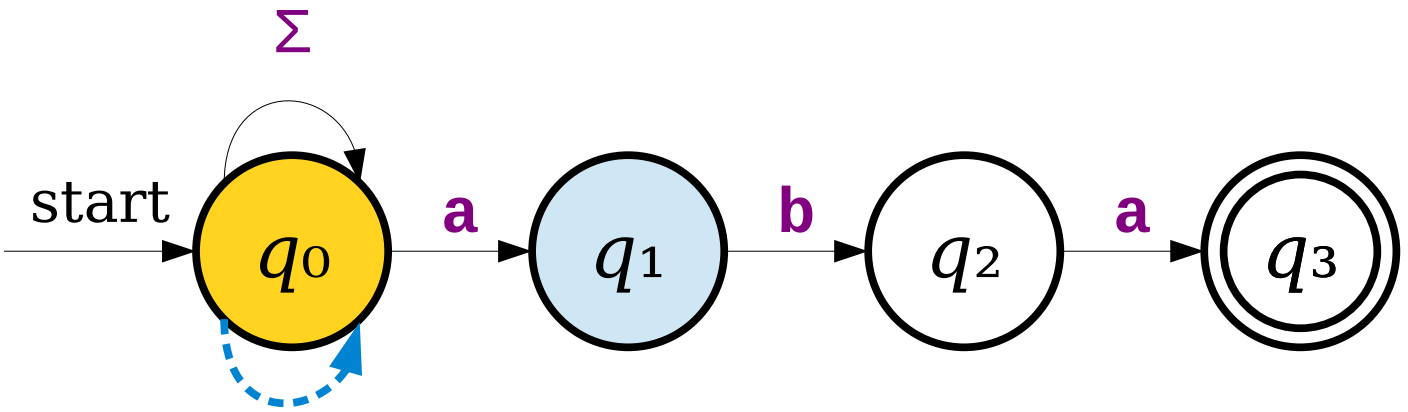

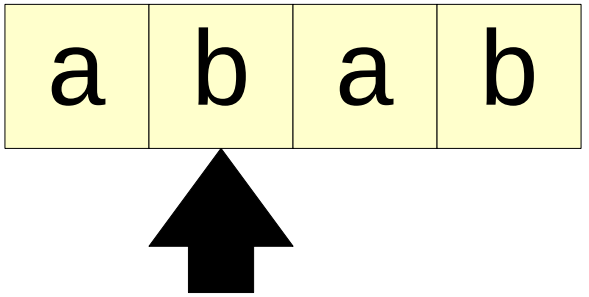

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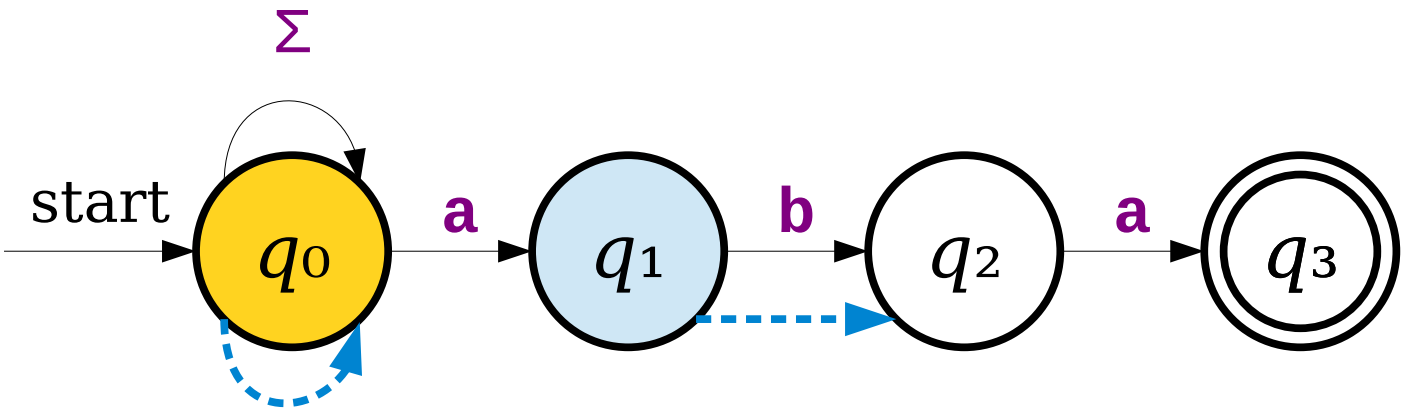

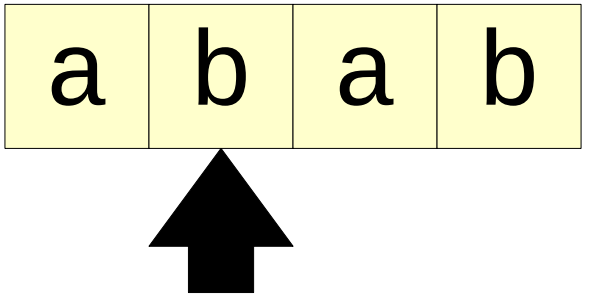

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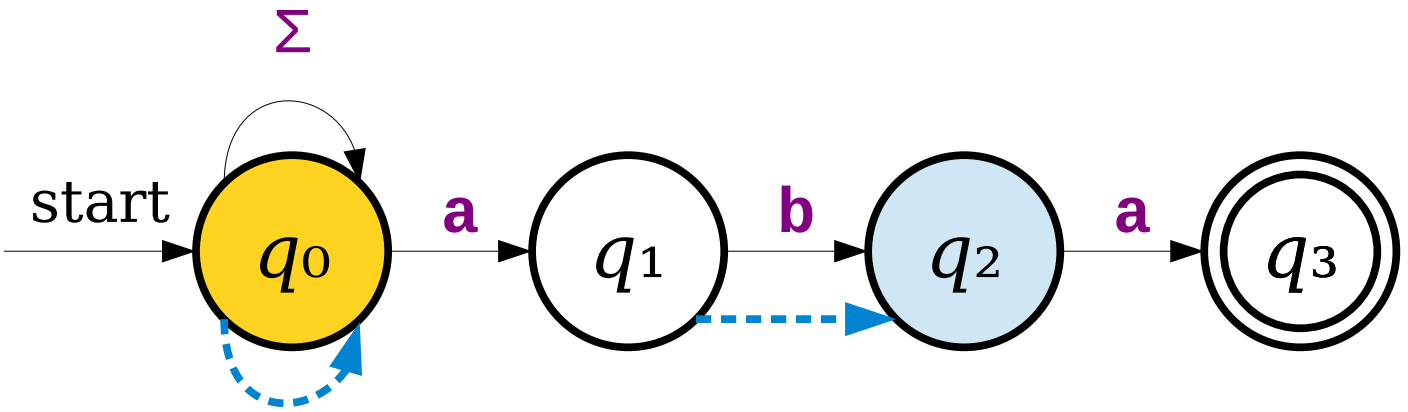

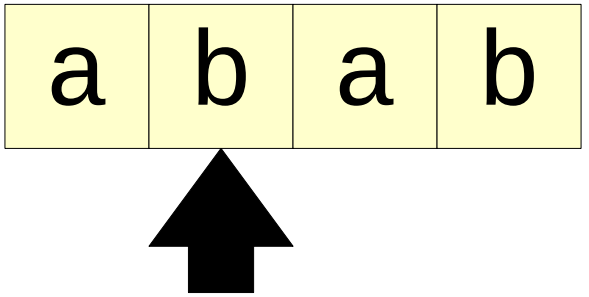

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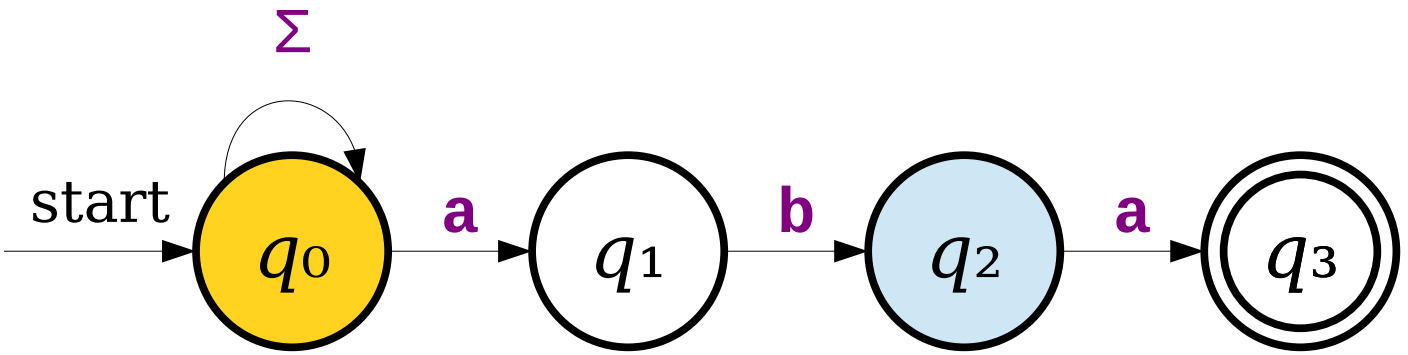

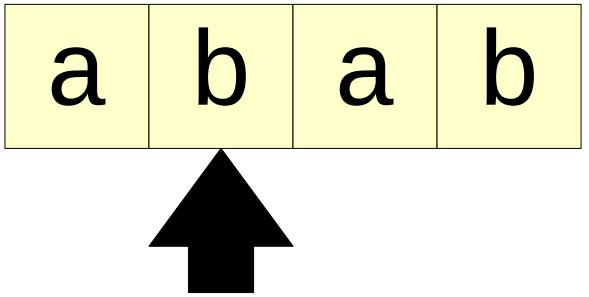

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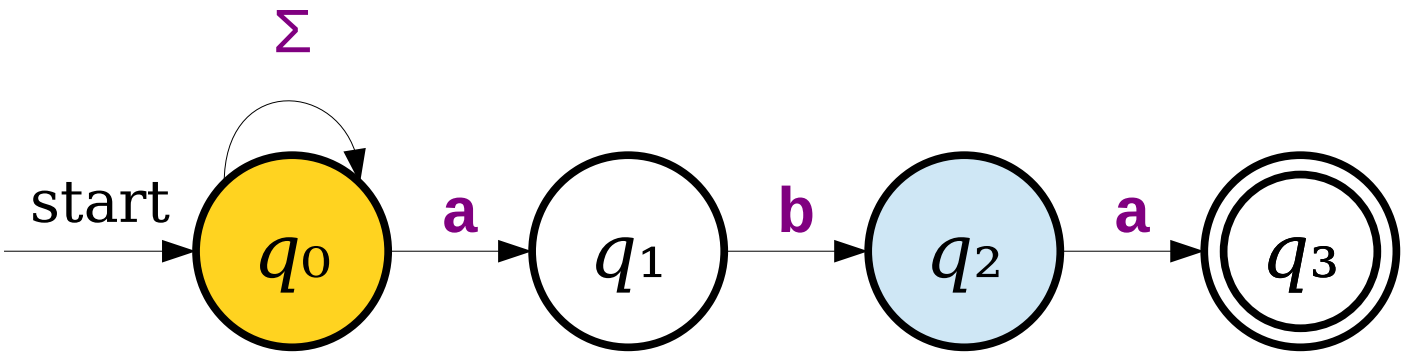

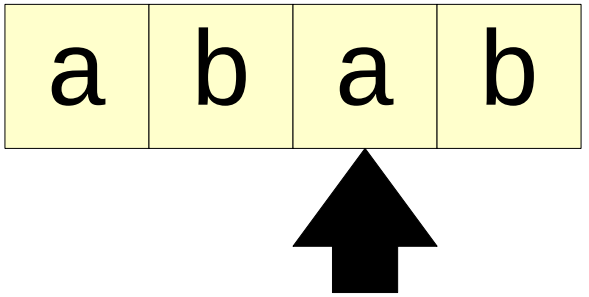

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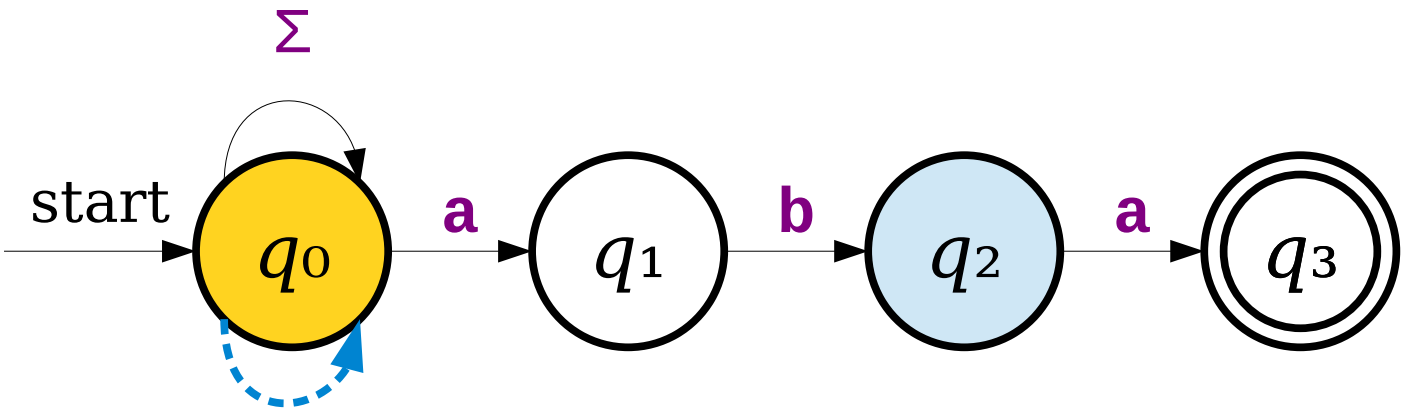

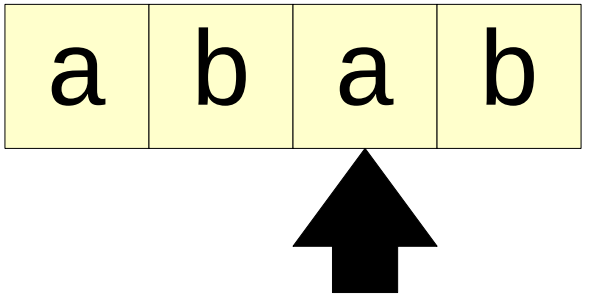

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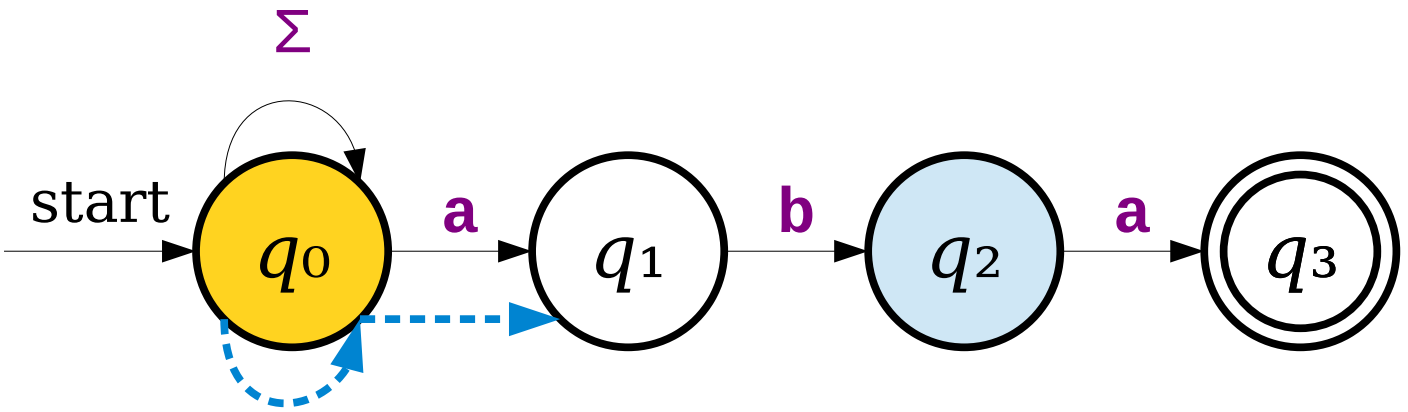

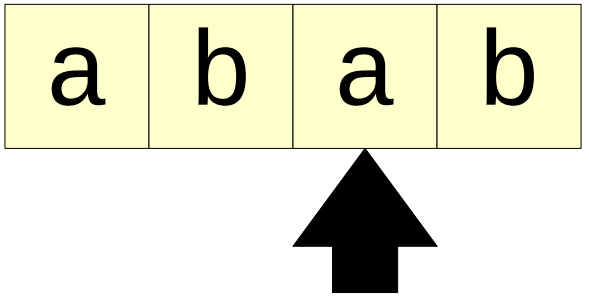

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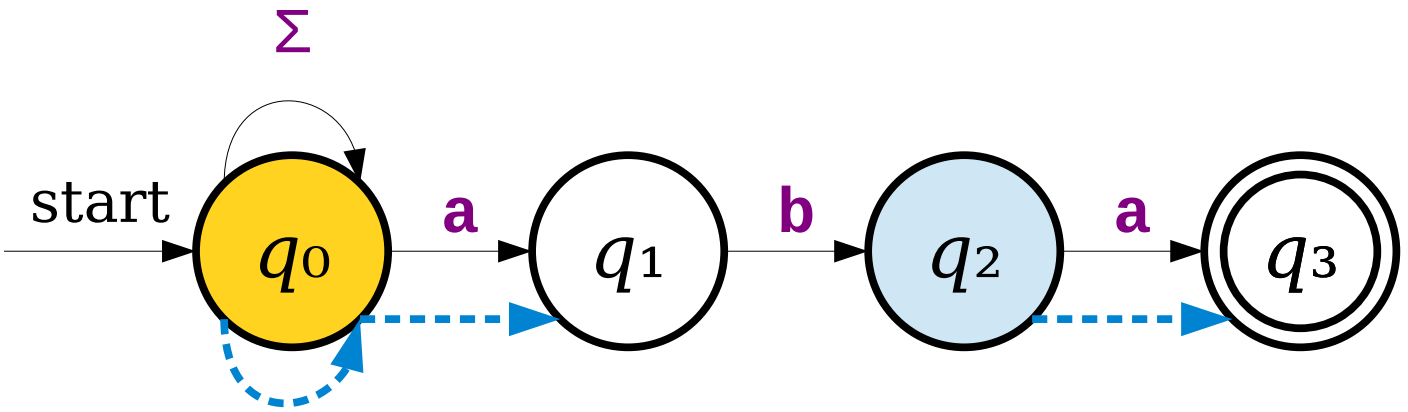

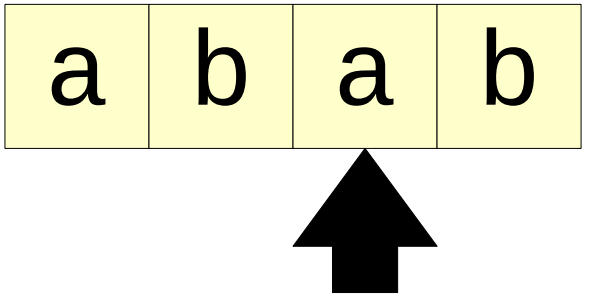

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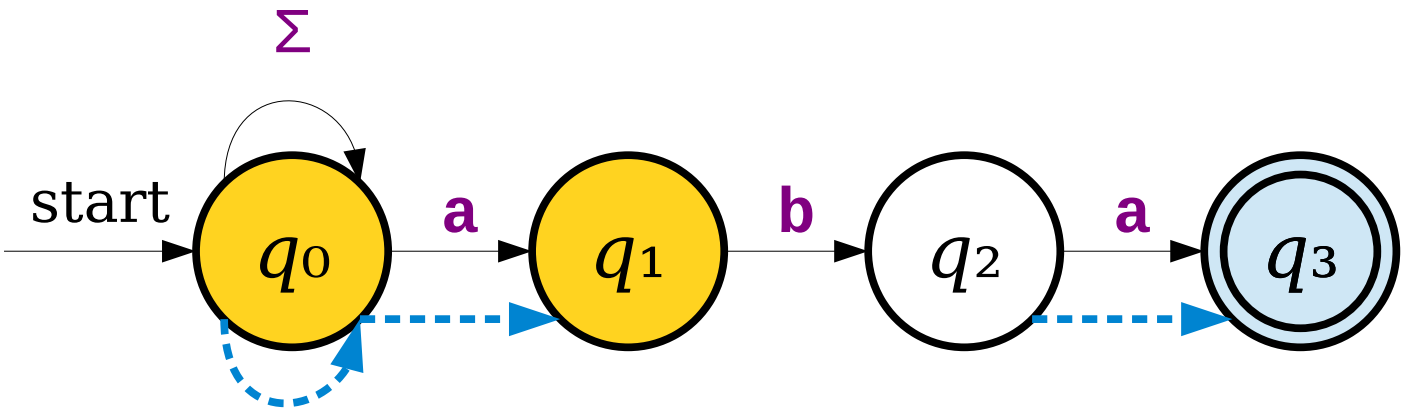

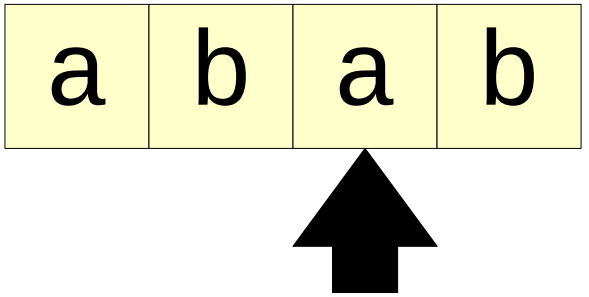

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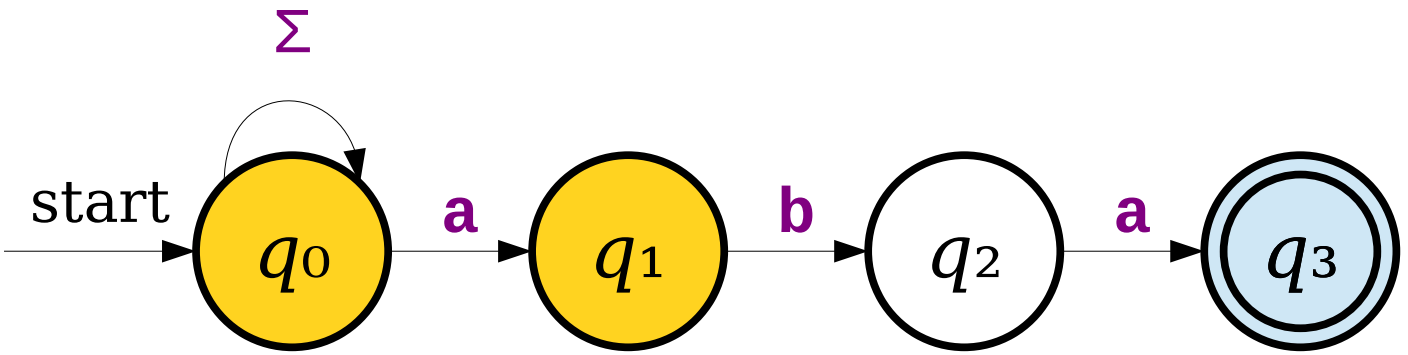

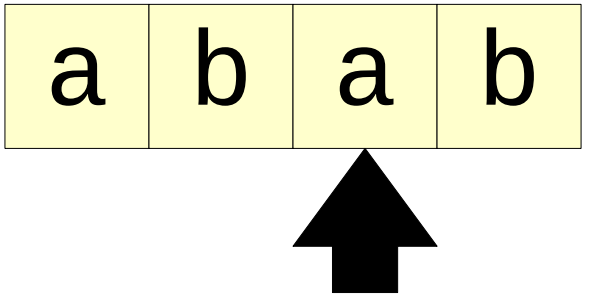

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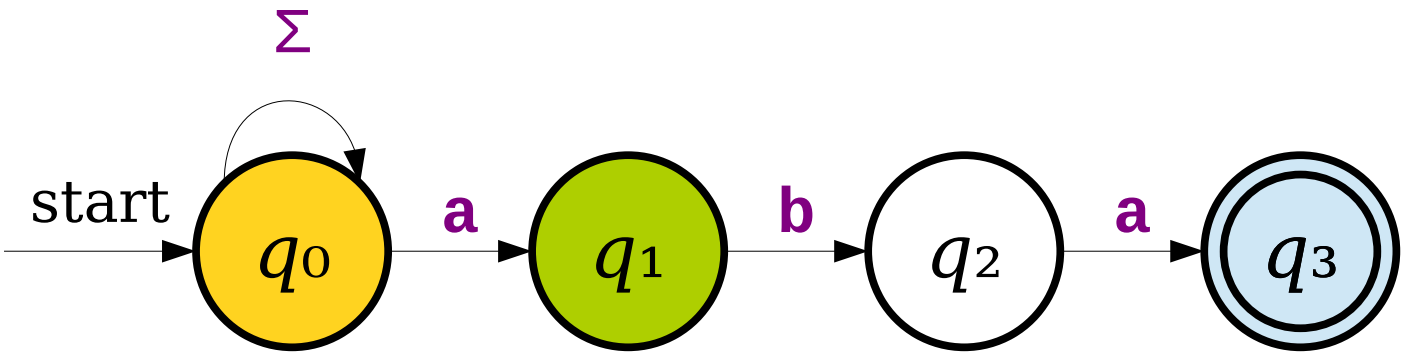
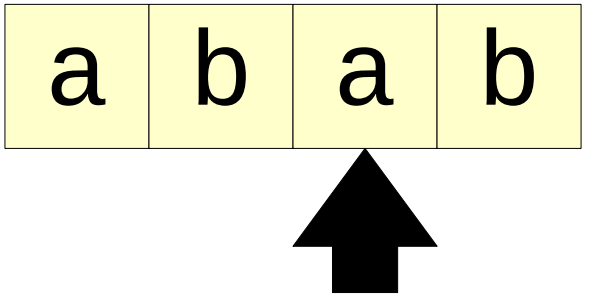
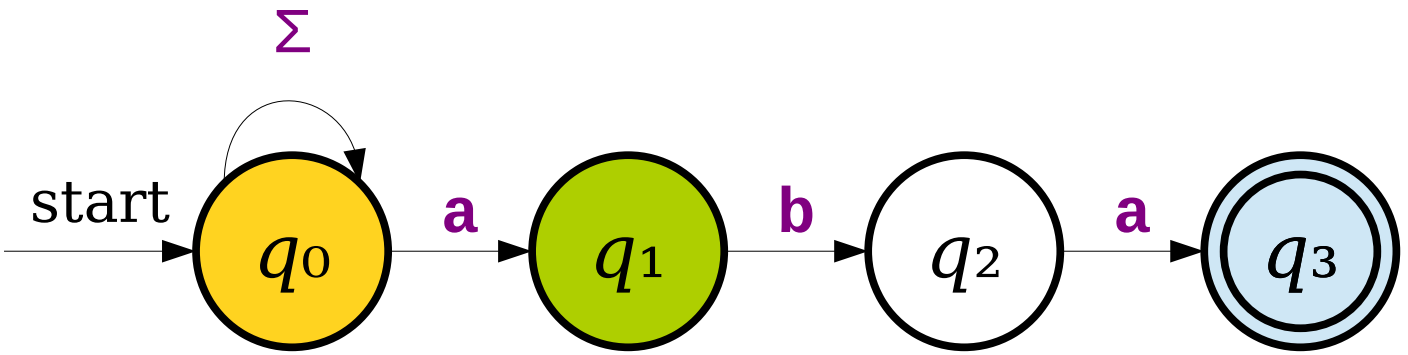
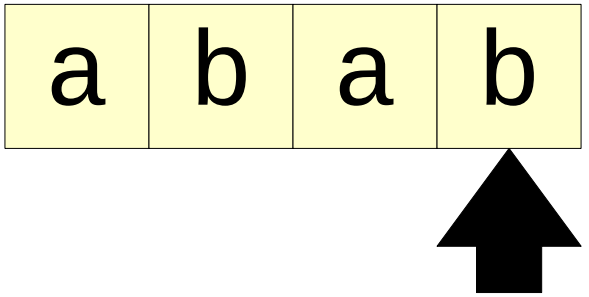




Massive Parallelism





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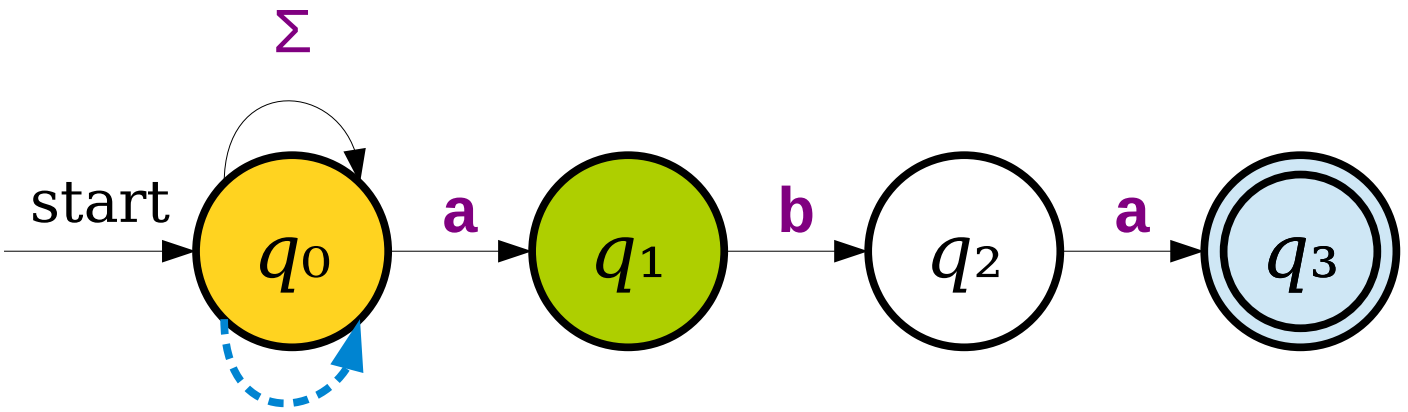

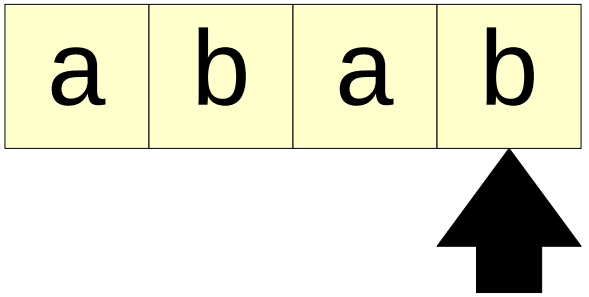

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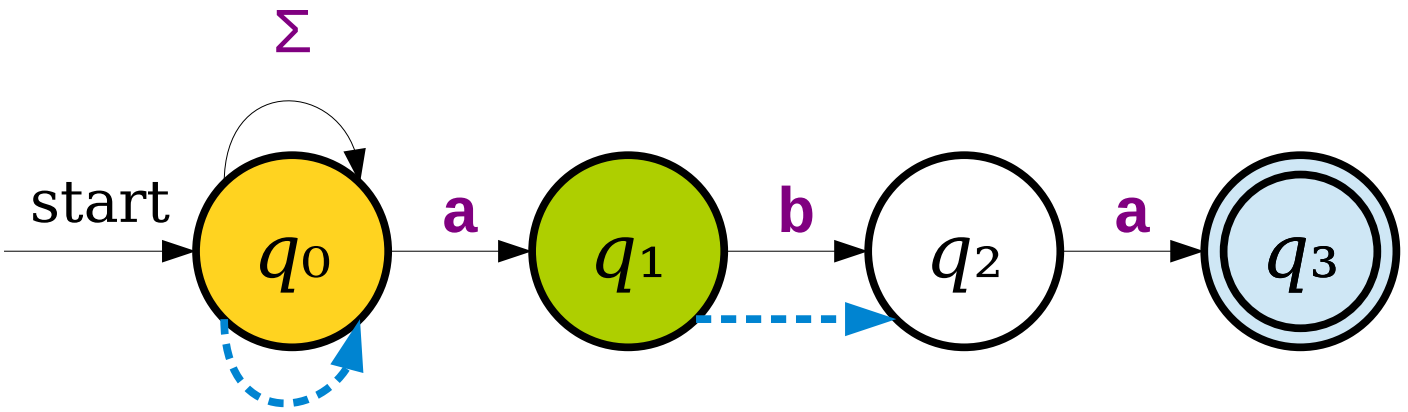

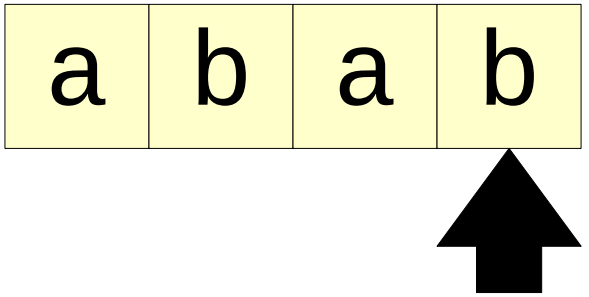

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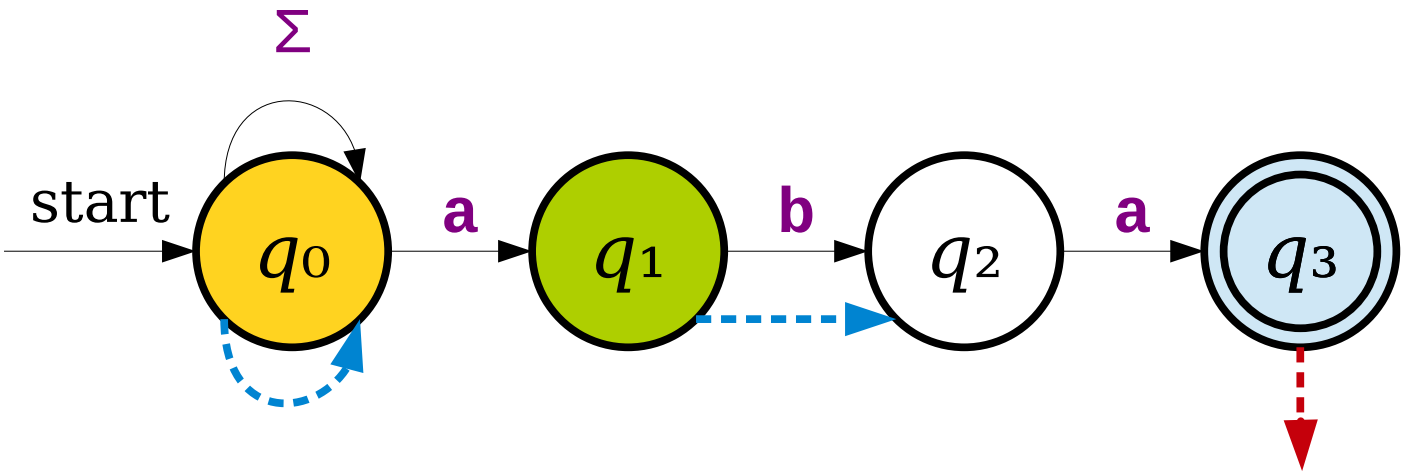

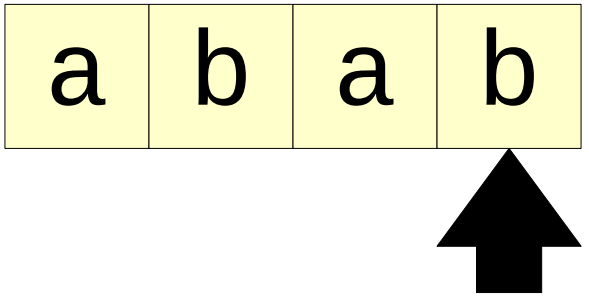

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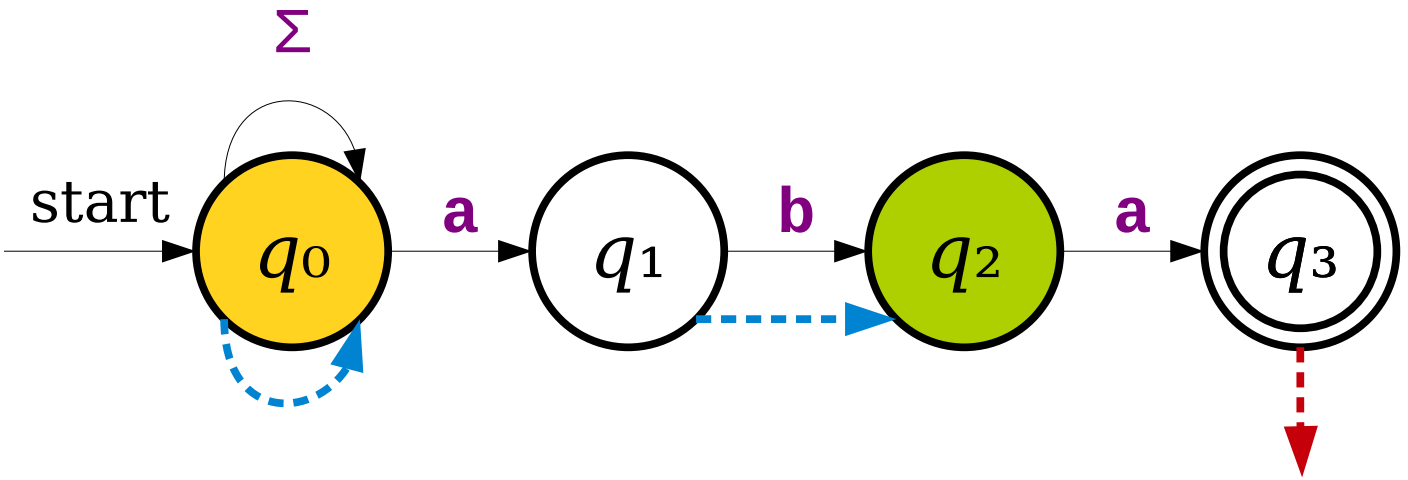

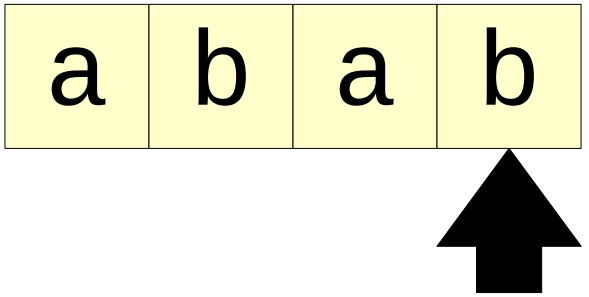

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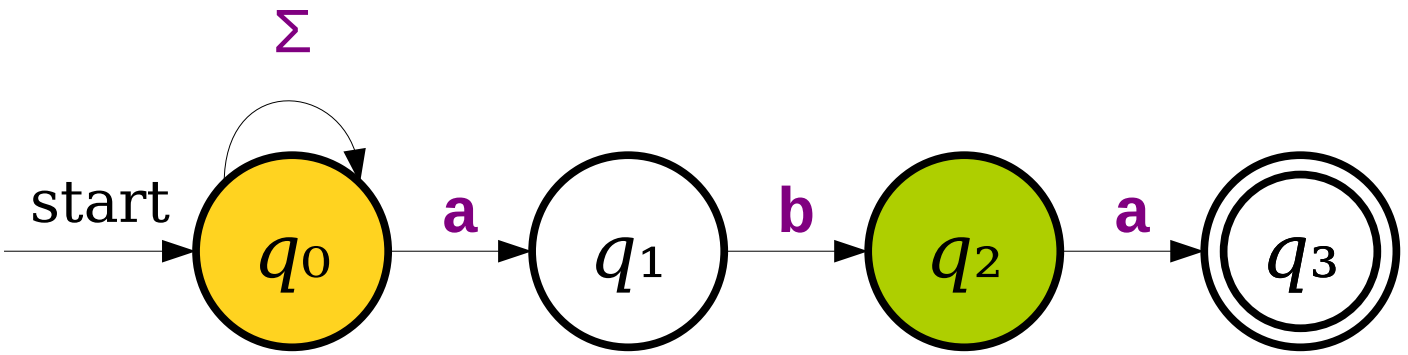
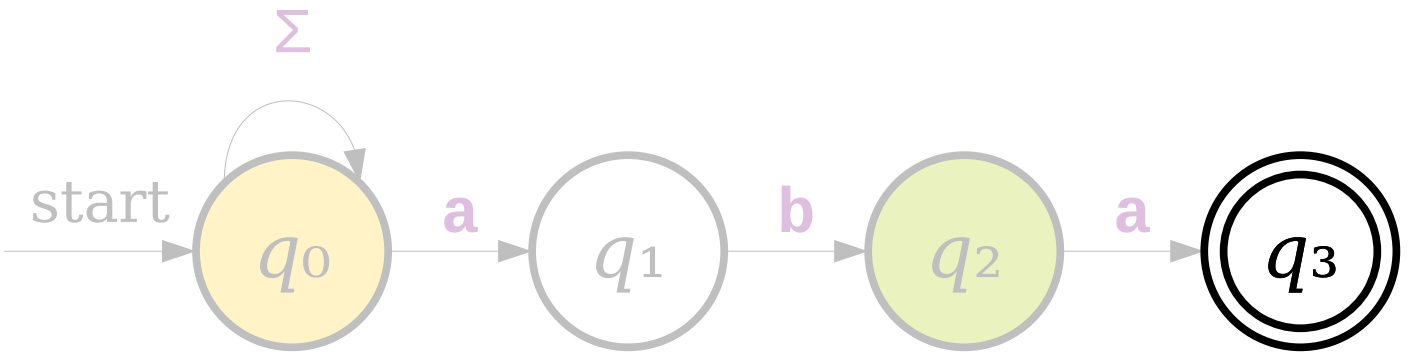


a	b	a	b

Massive Parallelism



We're not in any accepting state, so no possible path accepts.

a	b	a	b

Massive Parallelism

- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.
- (Here's a rigorous explanation about how this works; read this on your own time).
  - Start off in the set of all states formed by taking the start state and including each state that can be reached by zero or more ε-transitions.
  - When you read a symbol **a** in a set of states *S*:
    - Form the set *S'* of states that can be reached by following a single **a** transition from some state in *S*.
    - Your new set of states is the set of states in *S'*, plus the states reachable from *S'* by following zero or more ε-transitions.

Designing NFAs

- ***Embrace the nondeterminism!***
- Good model: ***Guess-and-check***:
  - Is there some information that you'd really like to have? Have the machine *nondeterministically guess* that information.
  - Then, have the machine *deterministically check* that the choice was correct.
- The *guess* phase corresponds to trying lots of different options.
- The *check* phase corresponds to filtering out bad guesses or wrong options.

Guess-and-Check

$L = \{ w \in \{\texttt{0}, \texttt{1}\}^* \mid w \text{ ends in } \texttt{010} \text{ or } \texttt{101} \}$

Guess-and-Check

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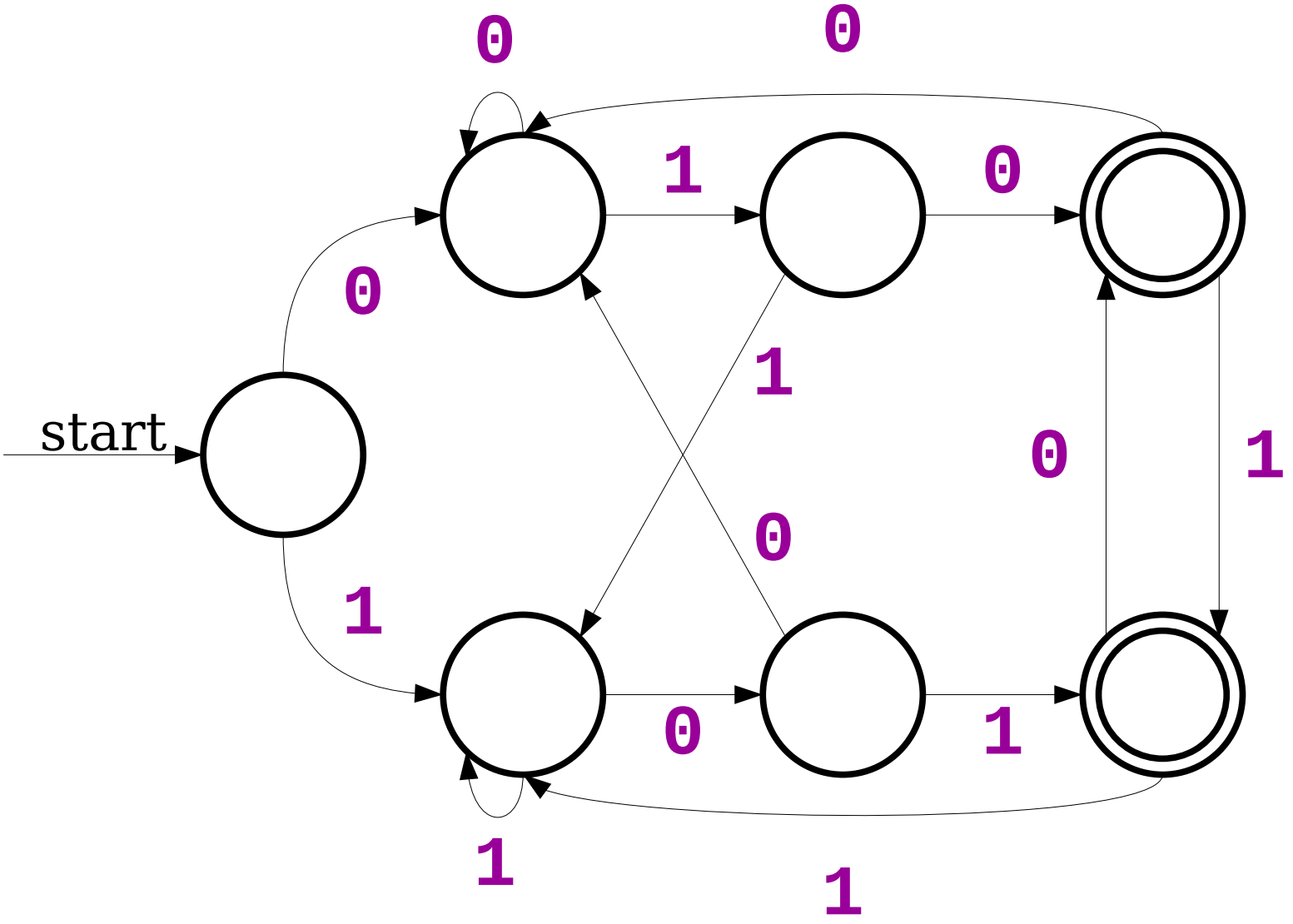

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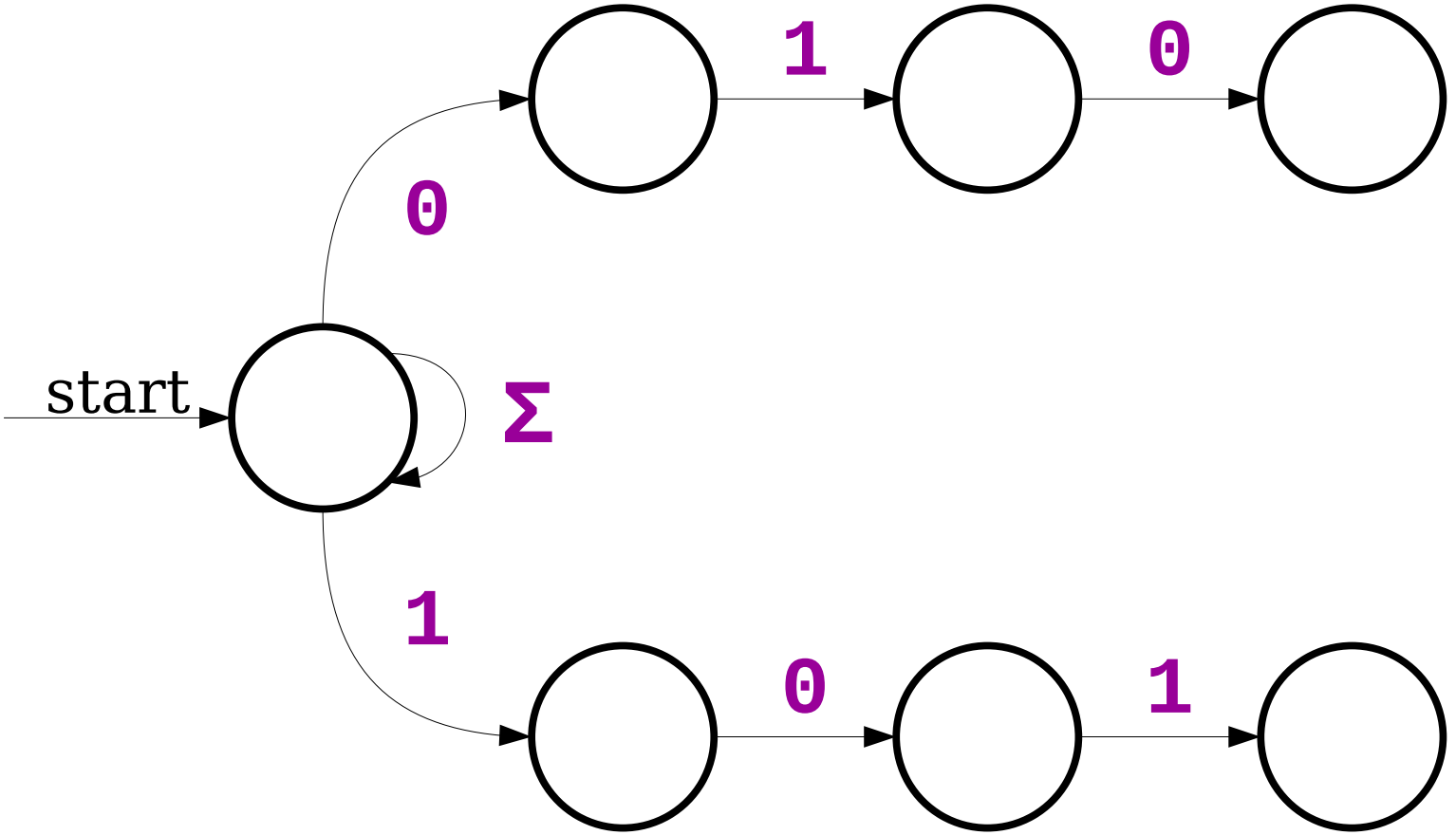

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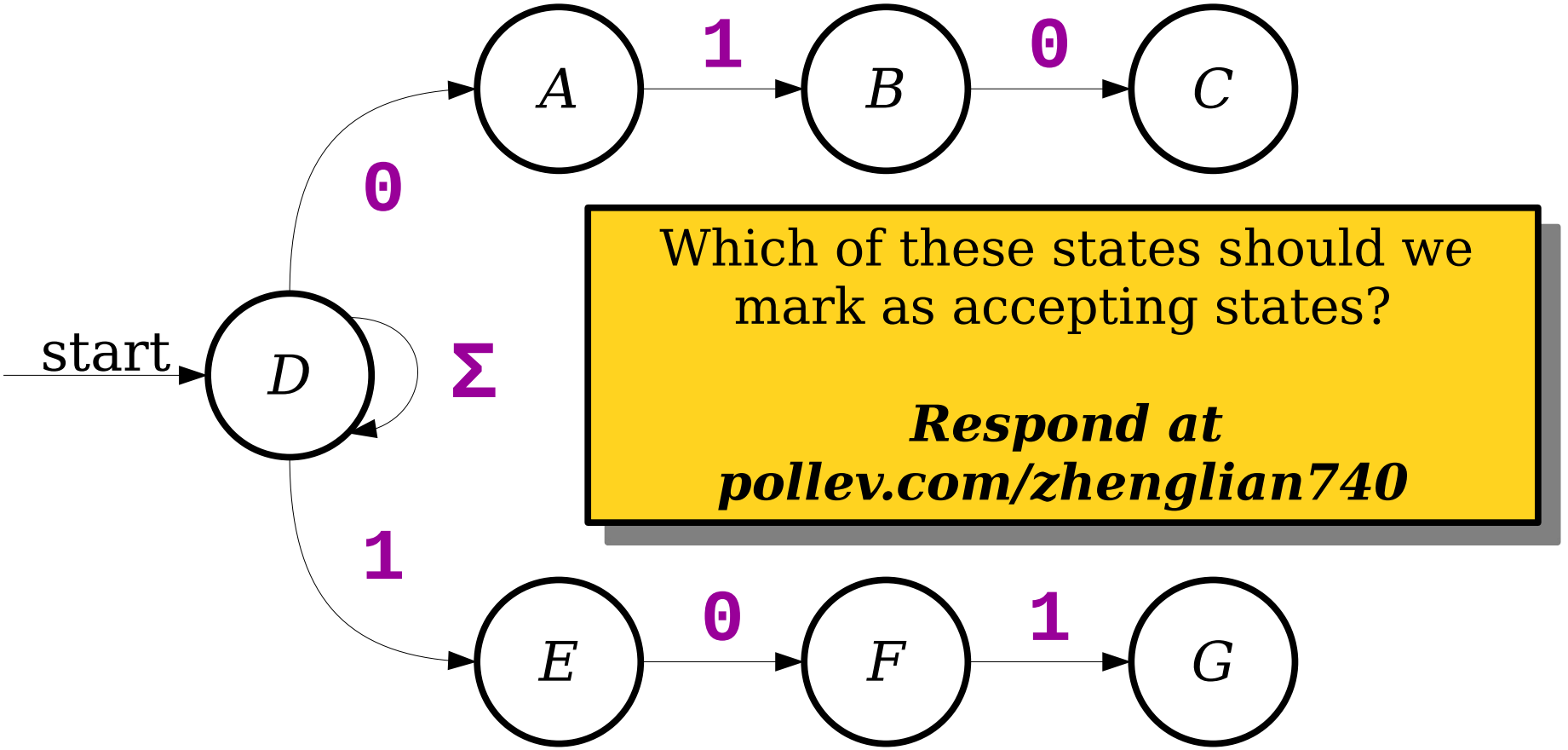

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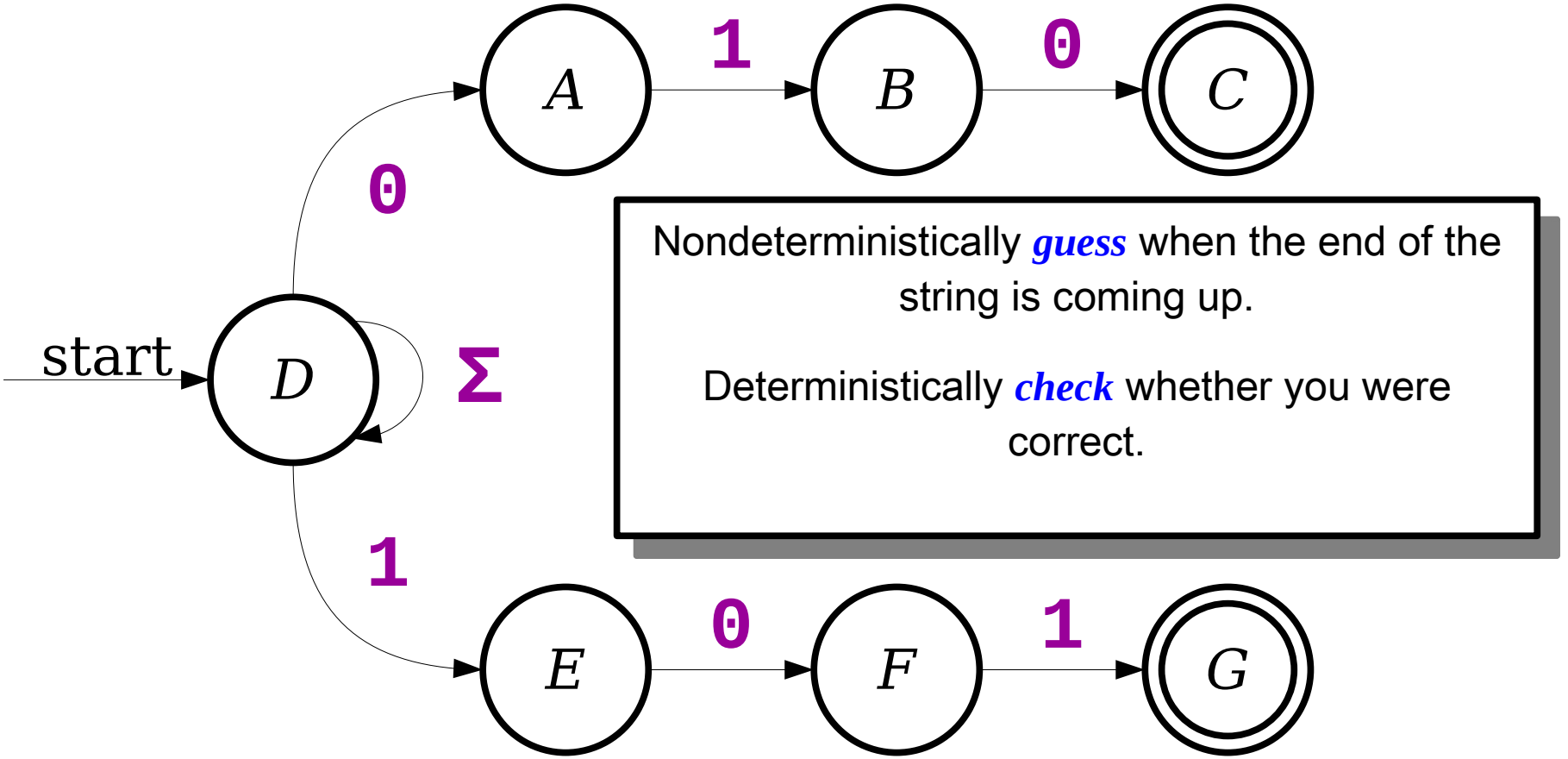

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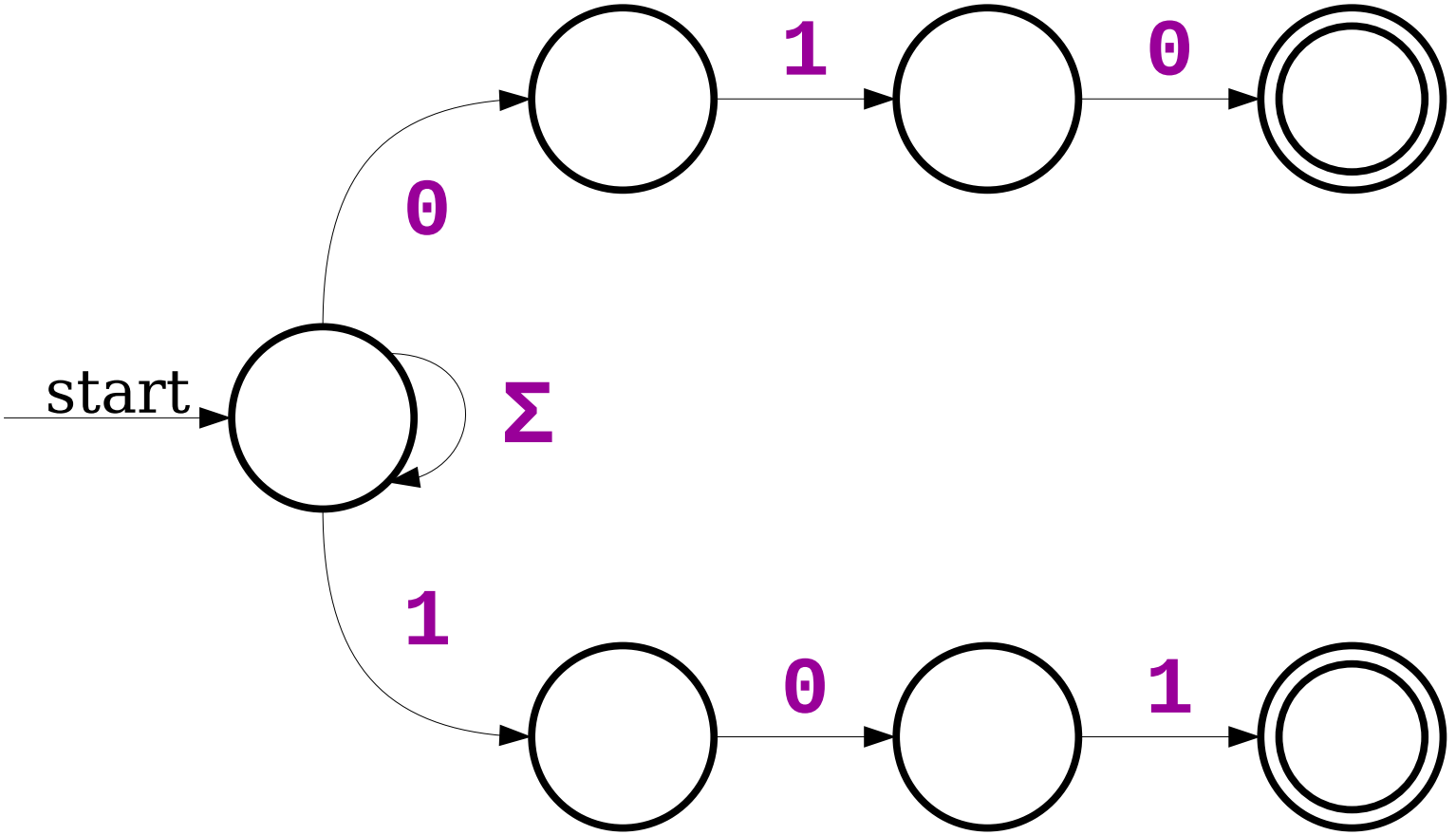

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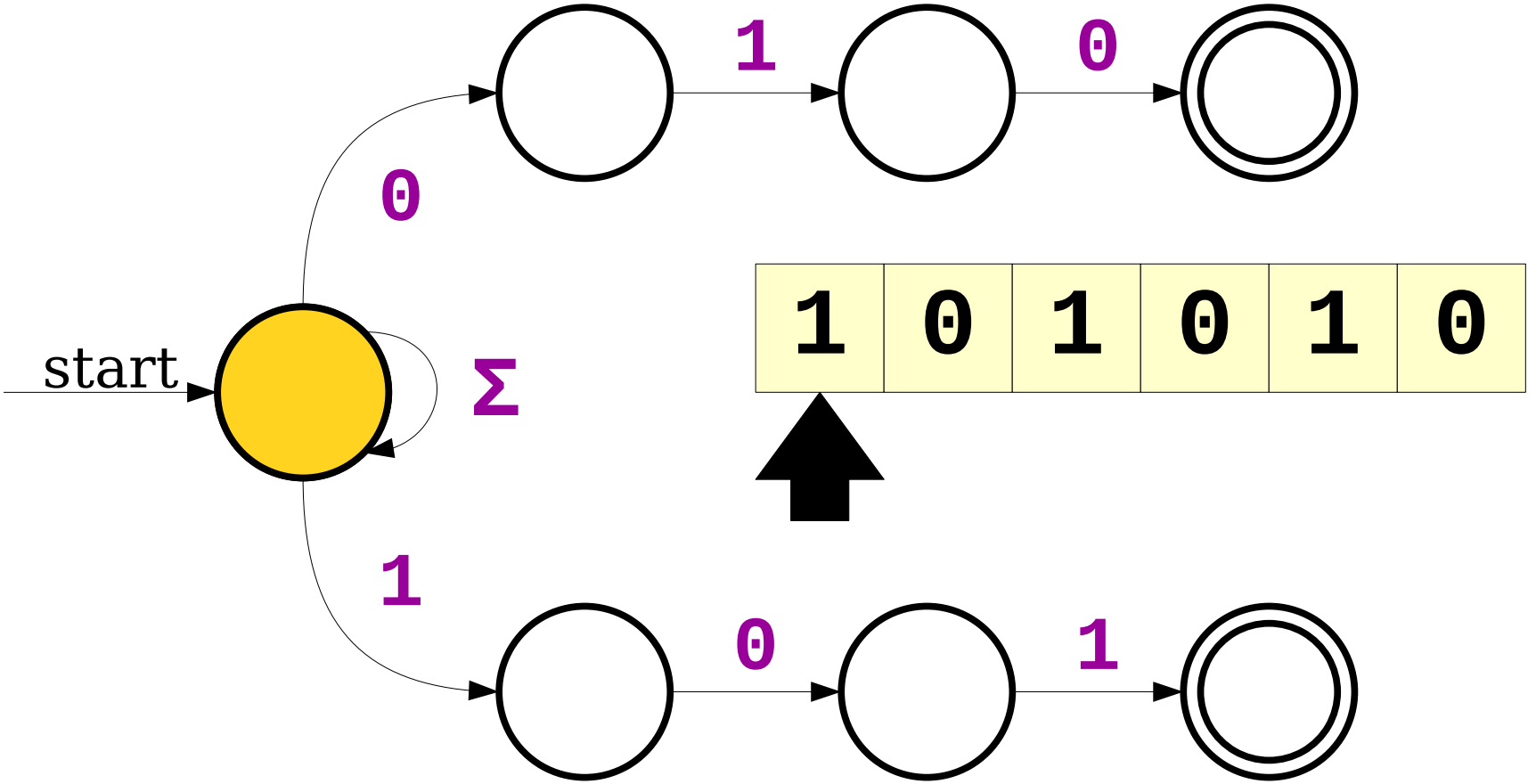

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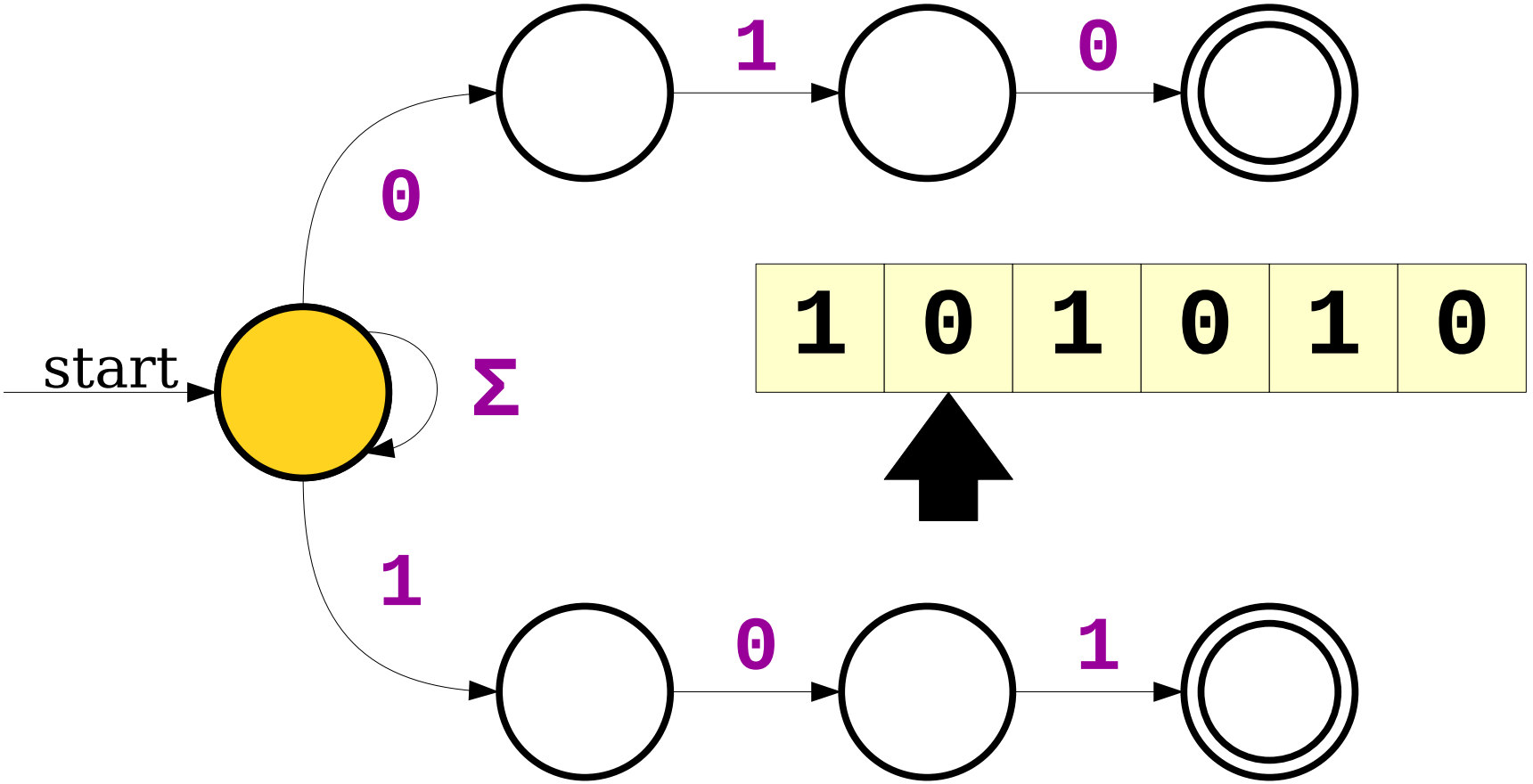

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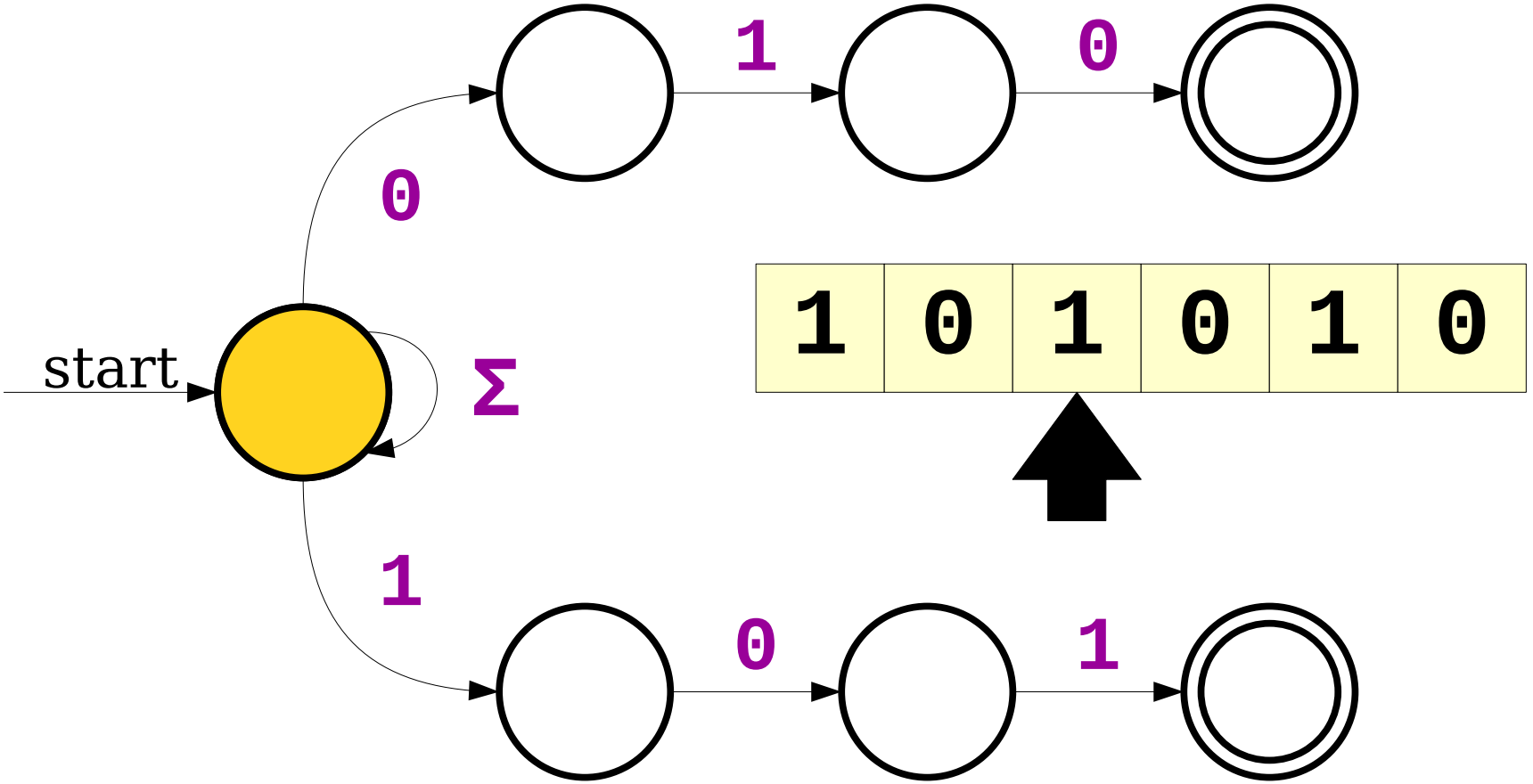

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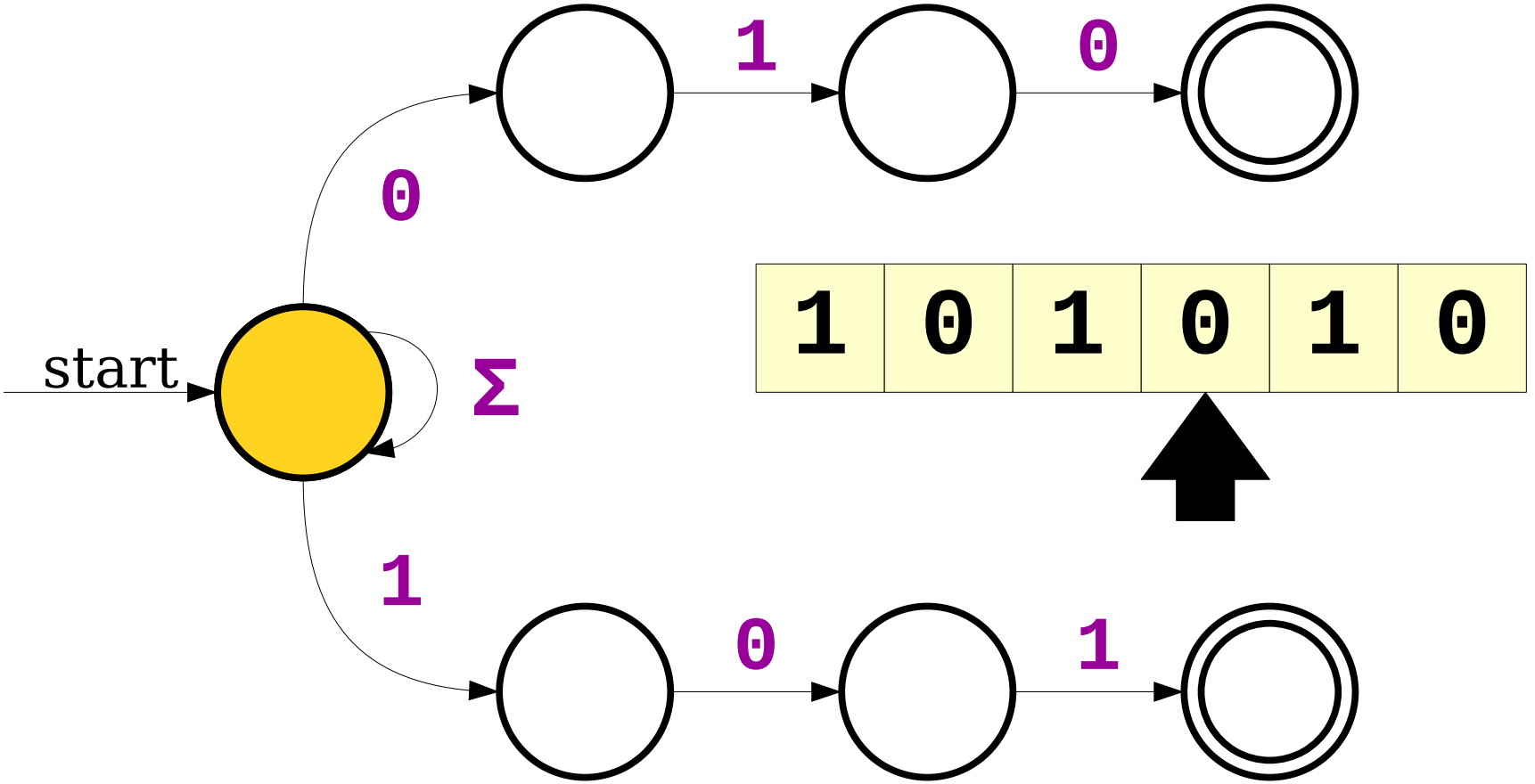

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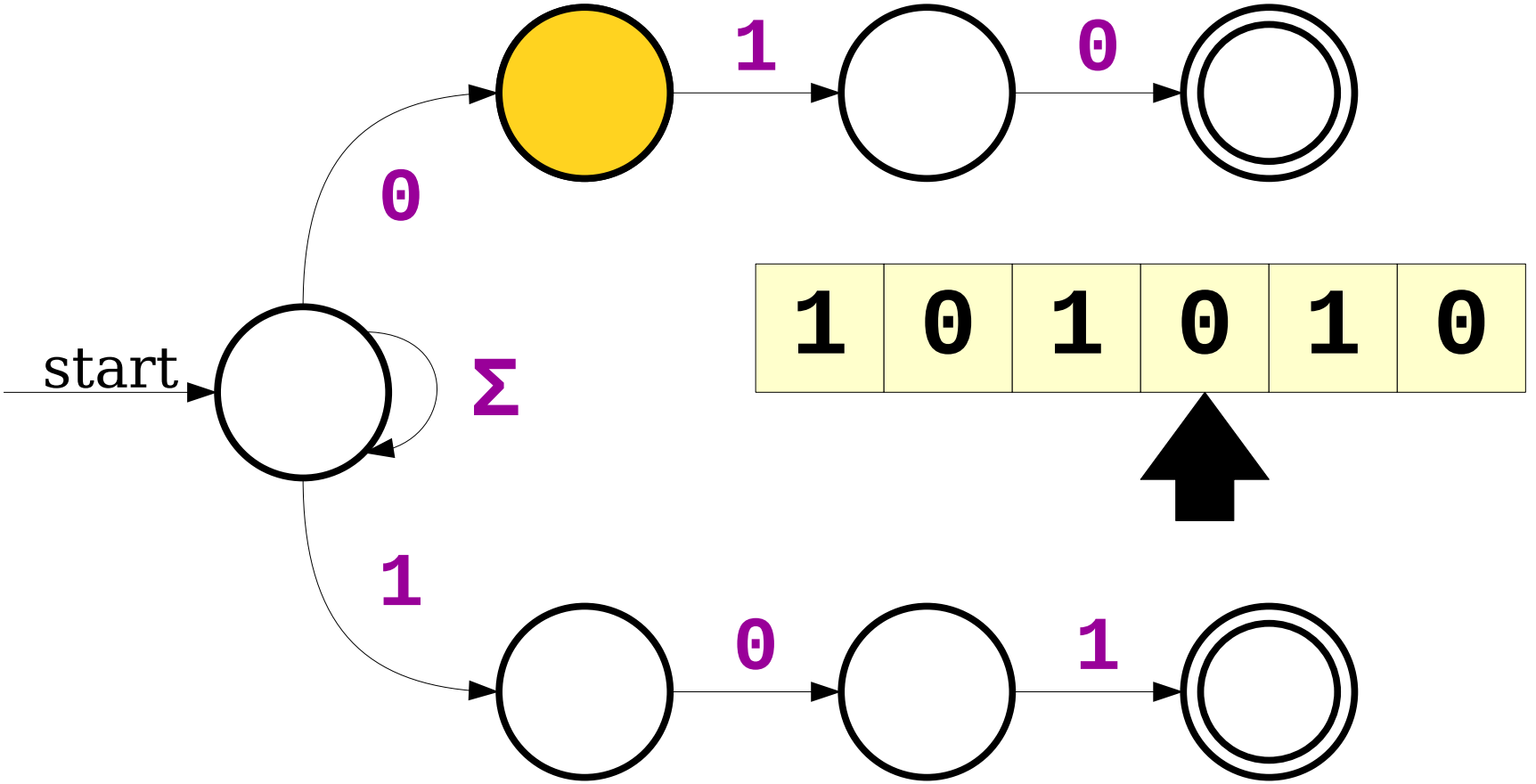

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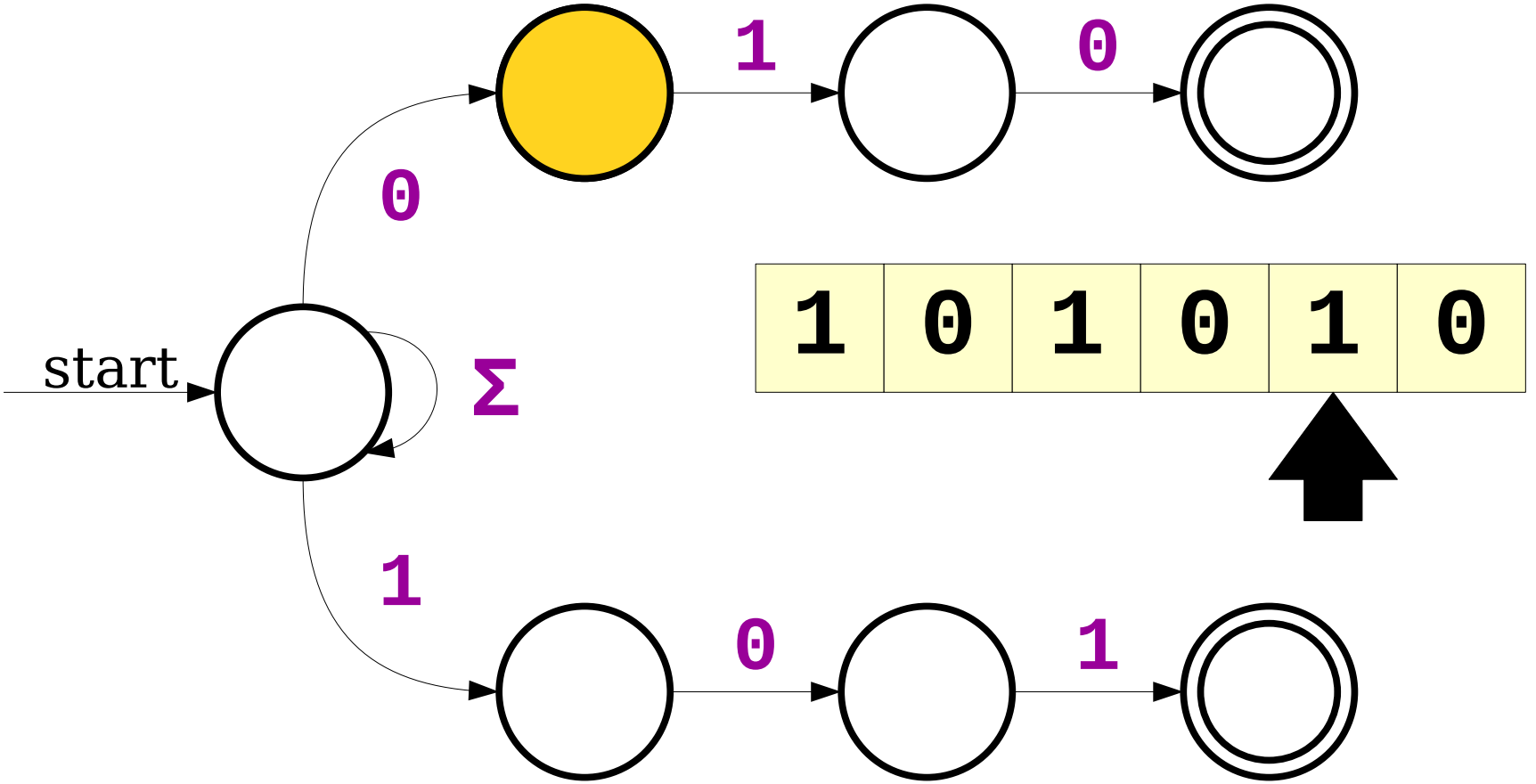

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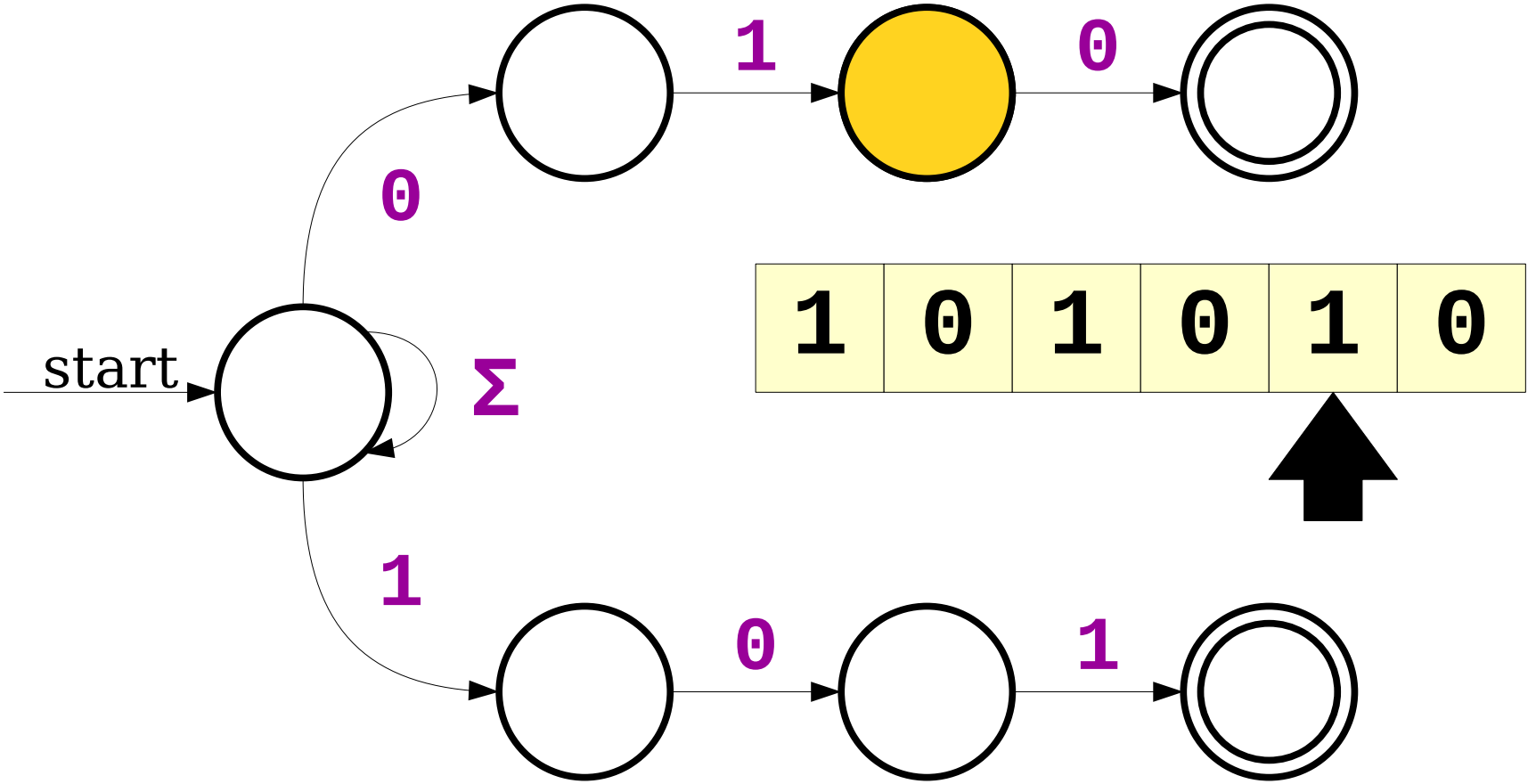

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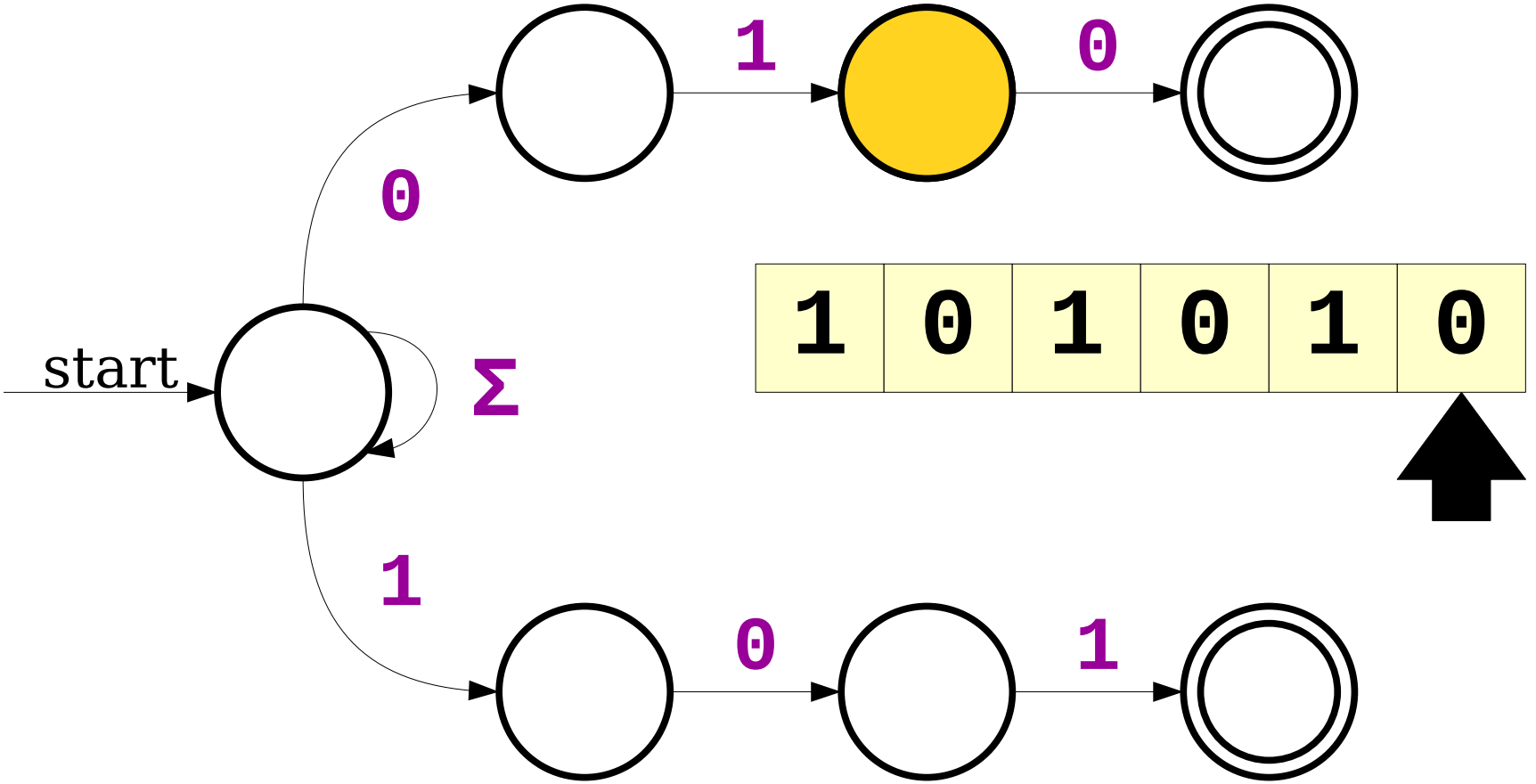

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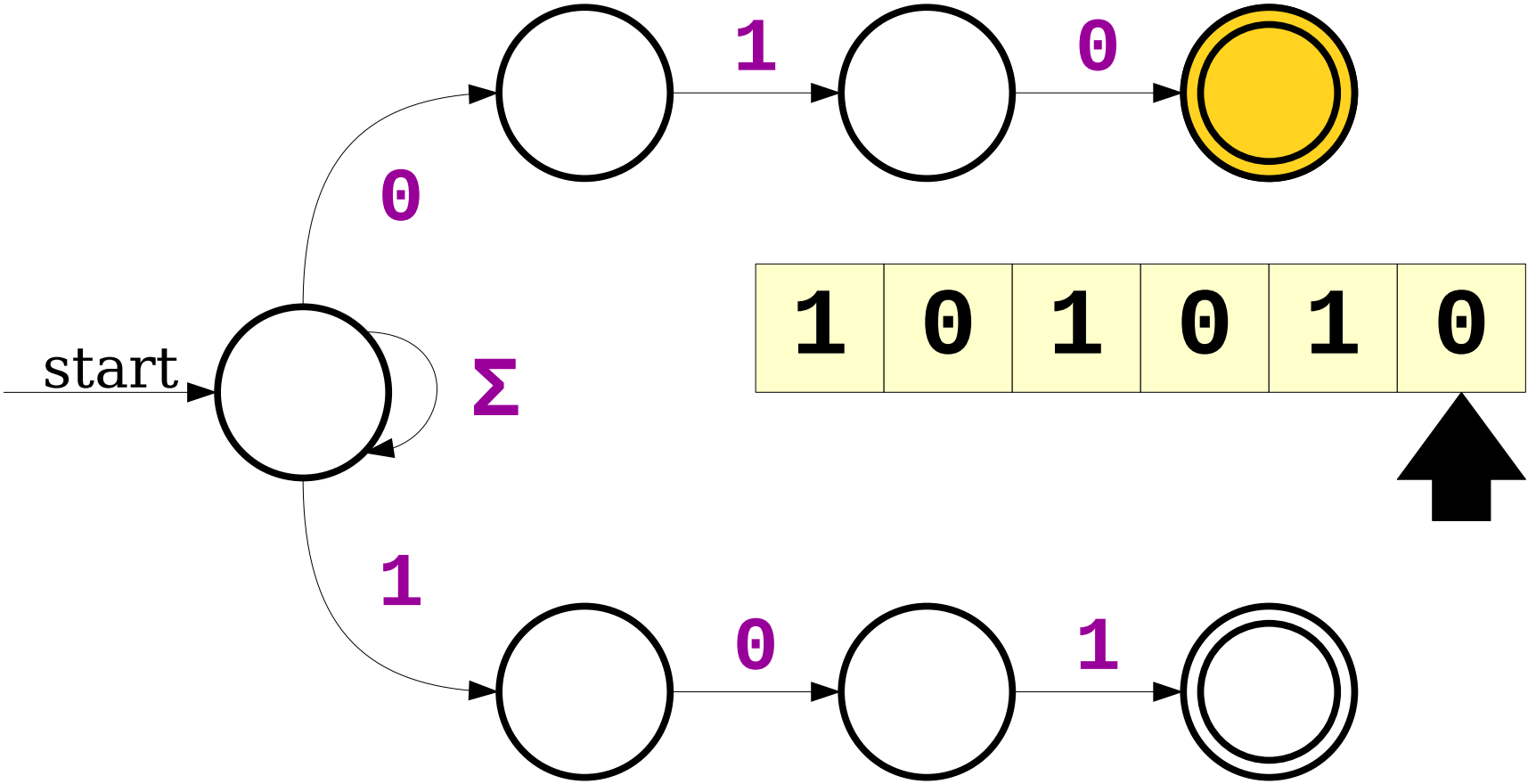

Guess-and-Check

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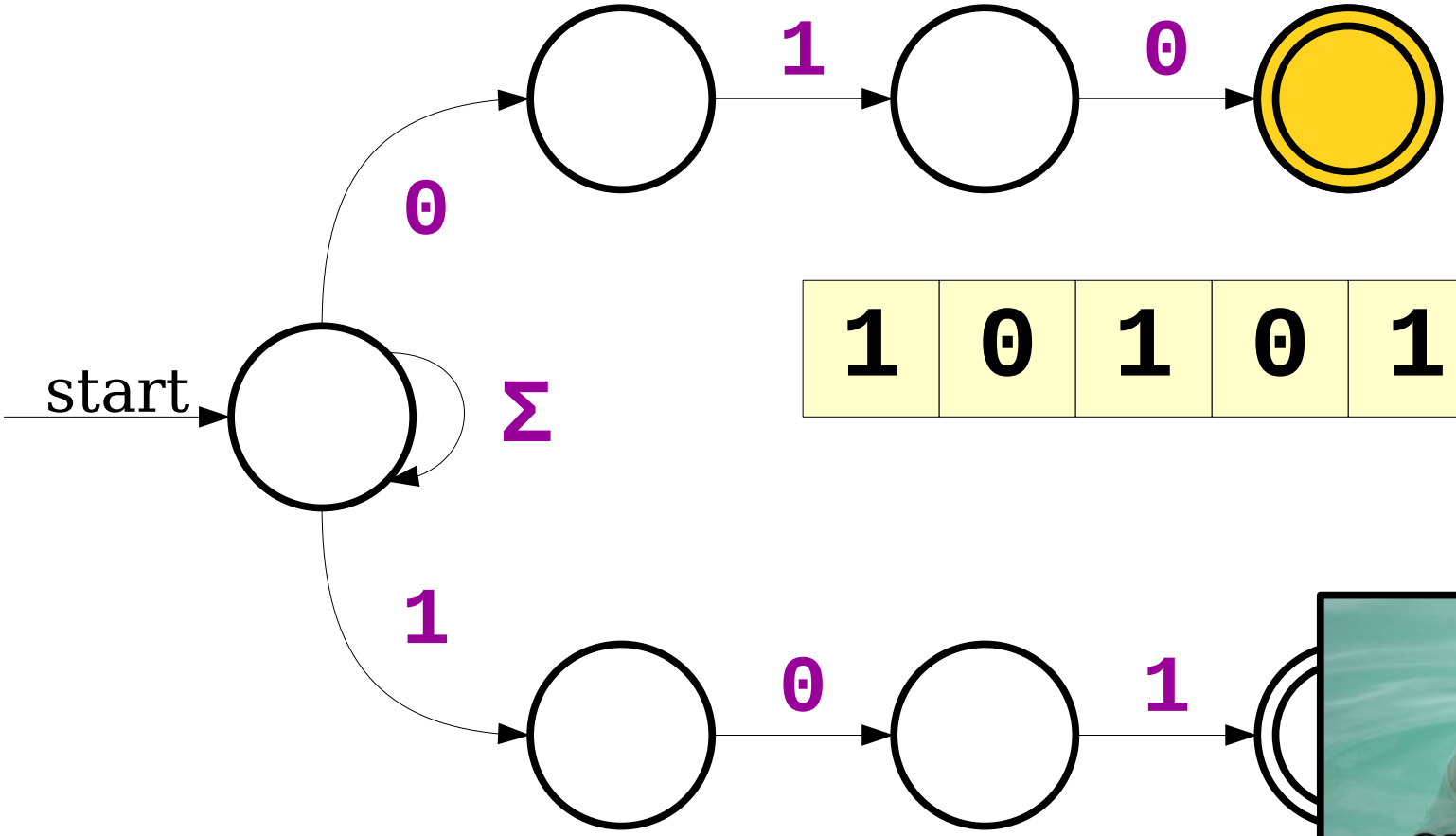

Guess-and-Check

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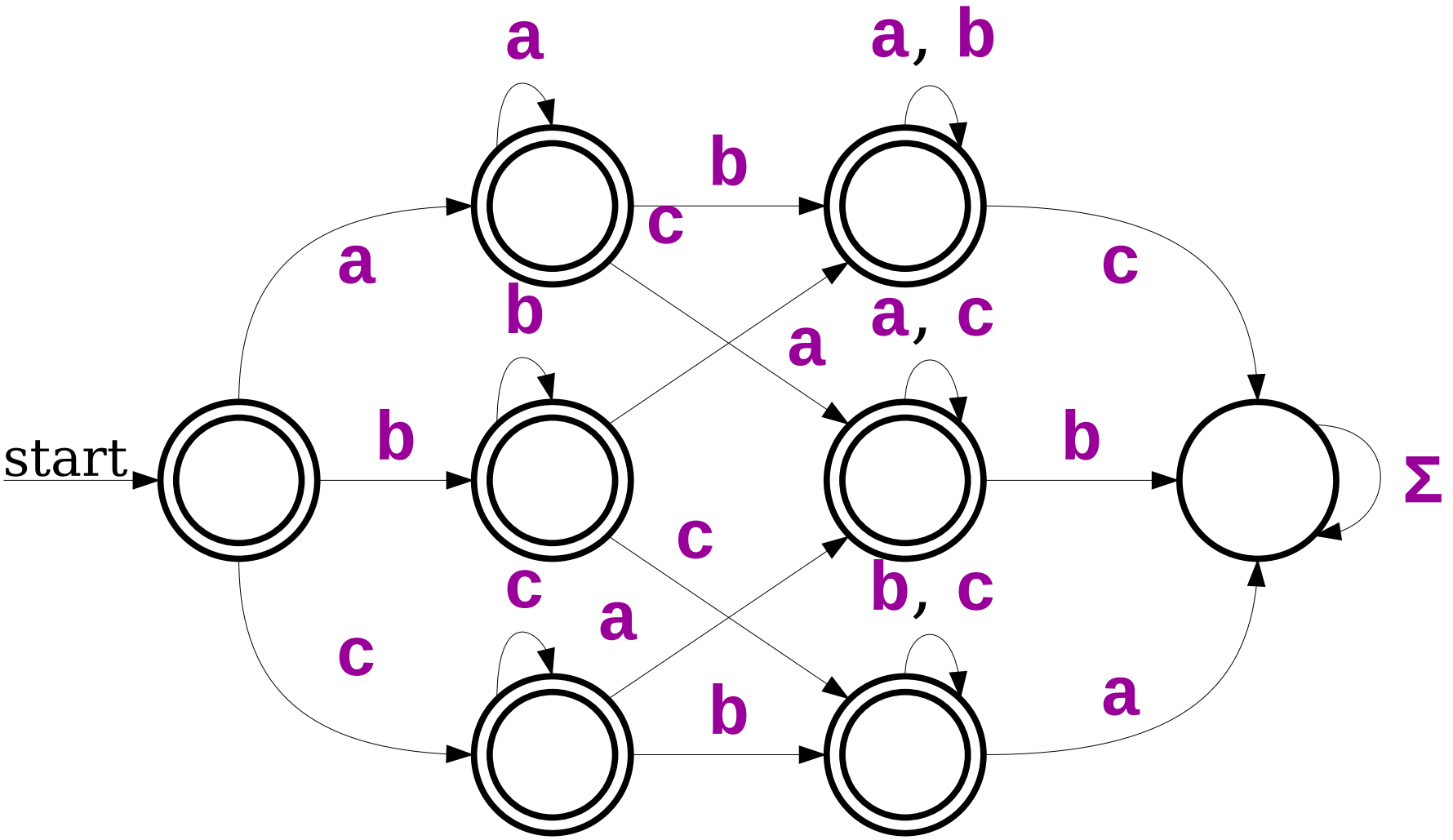

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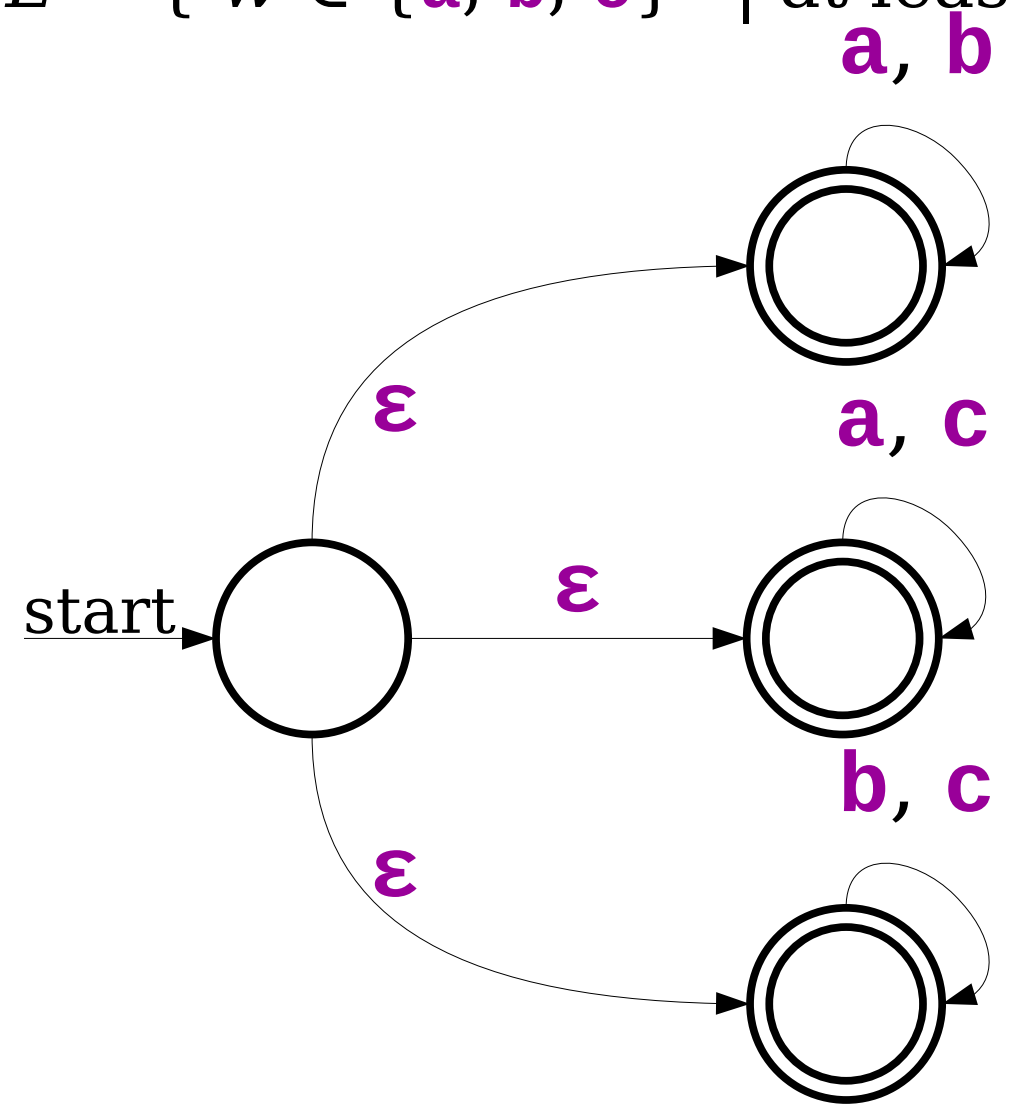
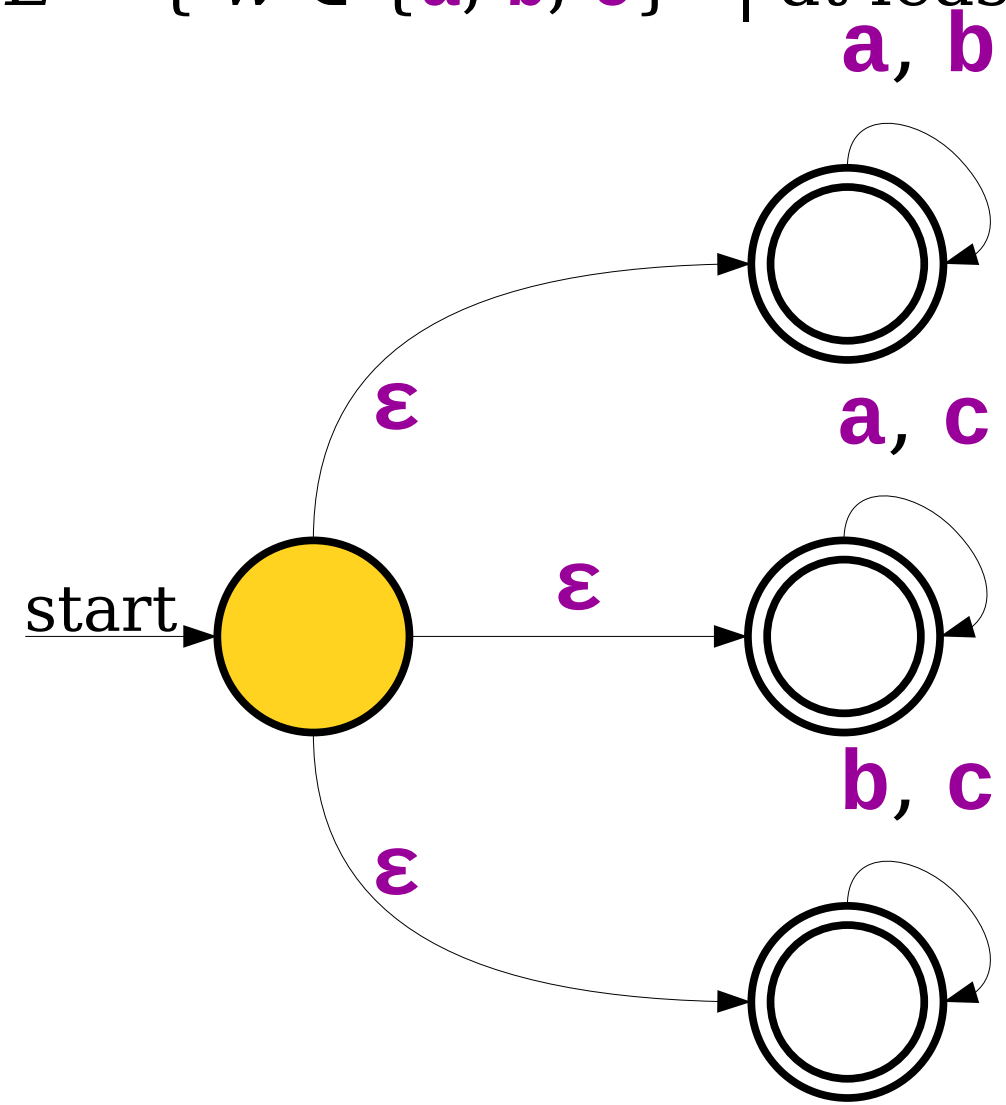


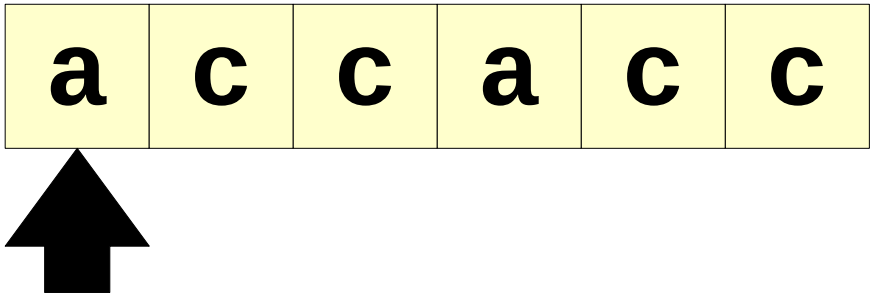
Nondeterministically ***guess*** which character is missing.

Deterministically ***check*** whether that character is indeed missing.

Guess-and-Check

$L = \{ w \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$





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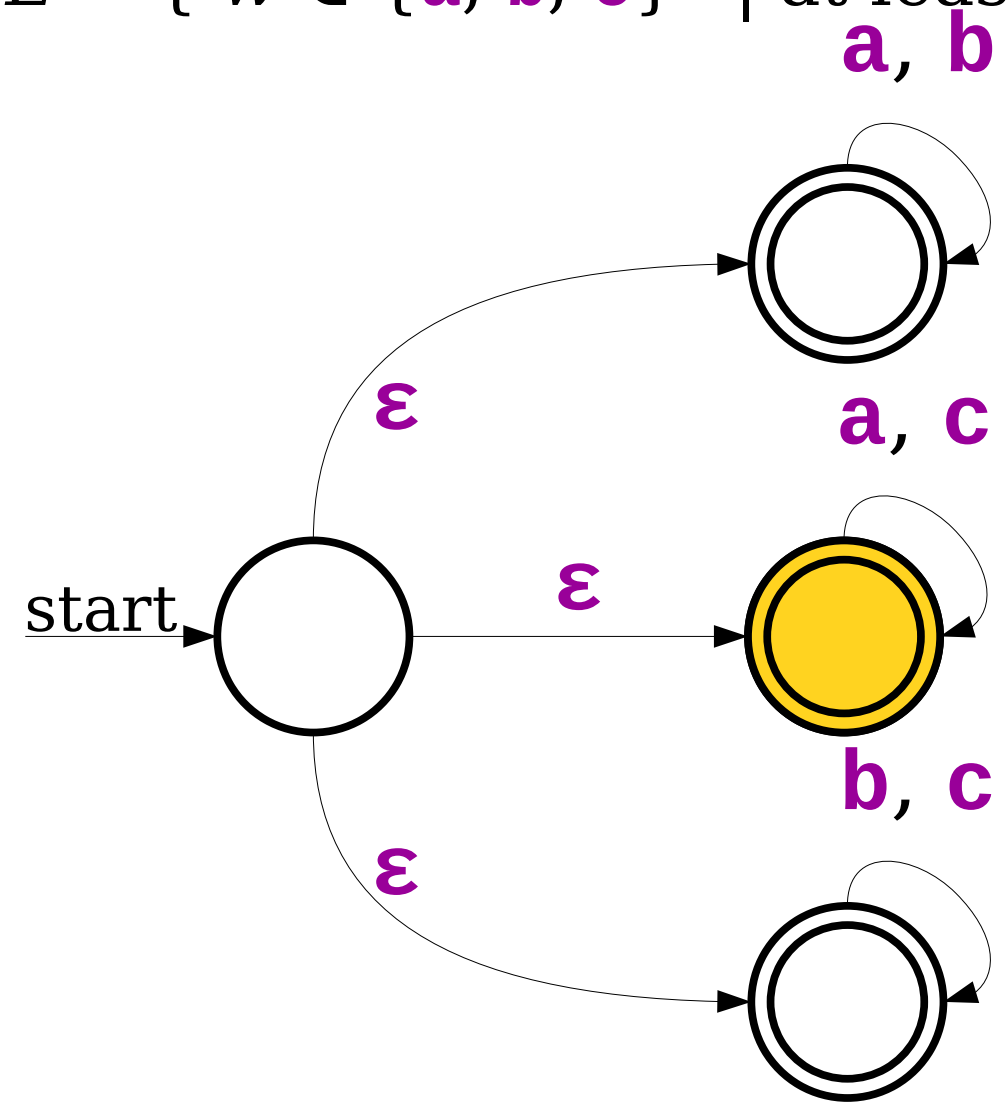

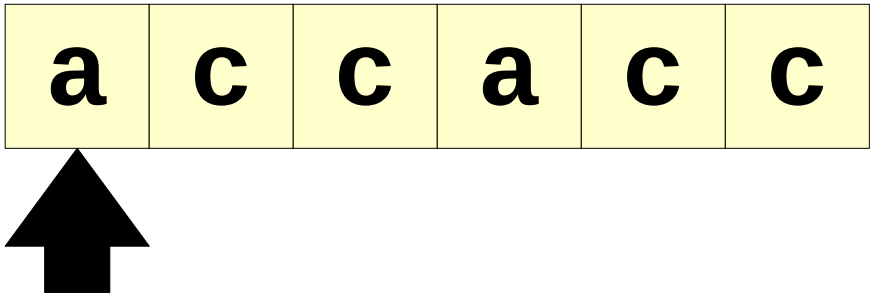

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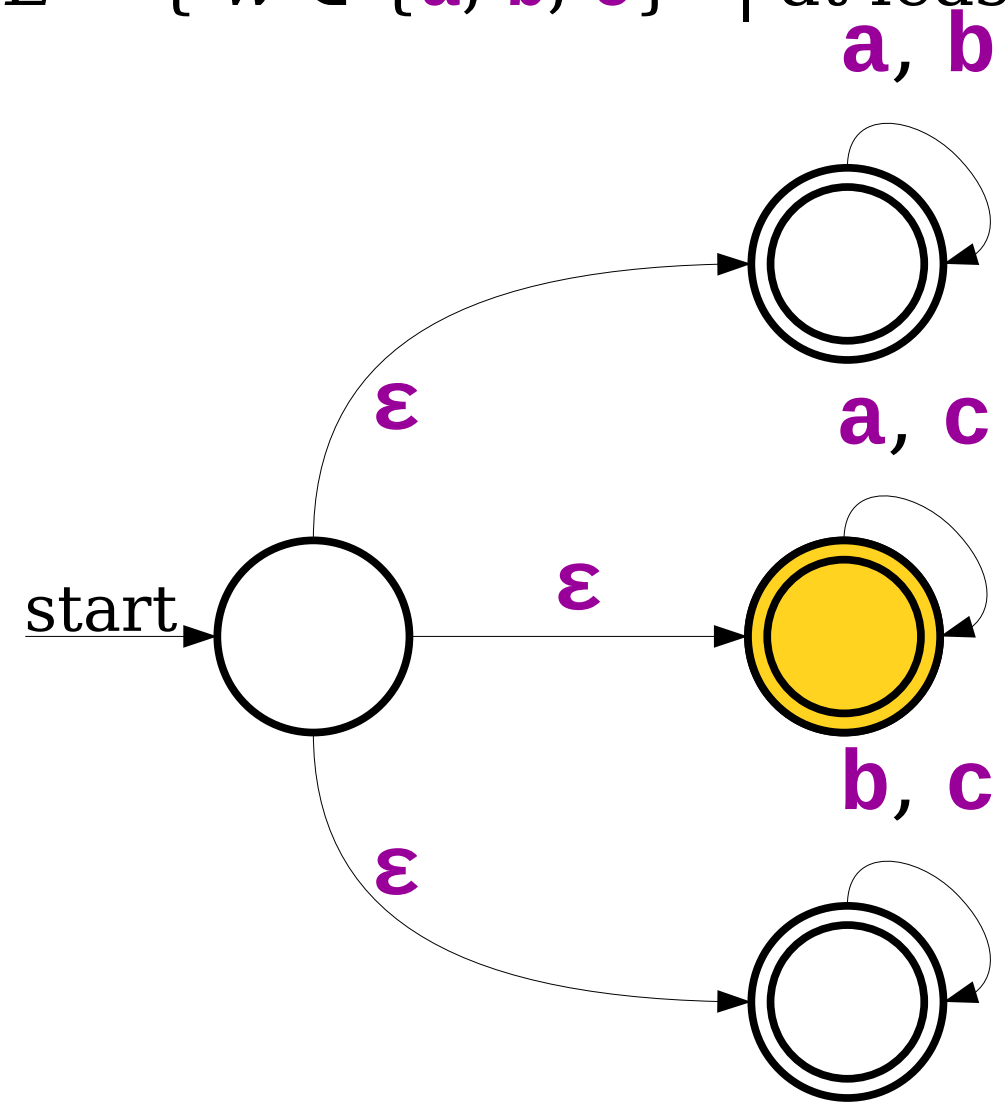

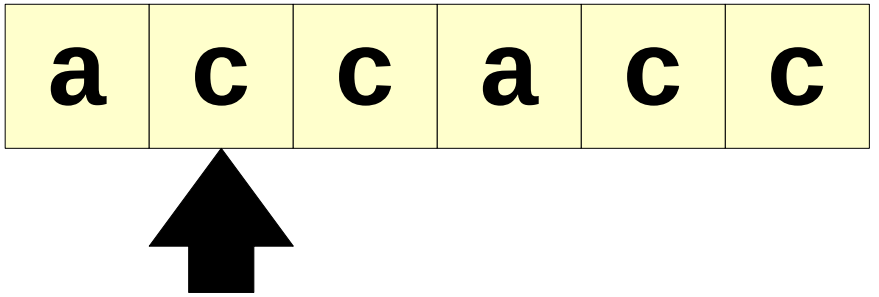

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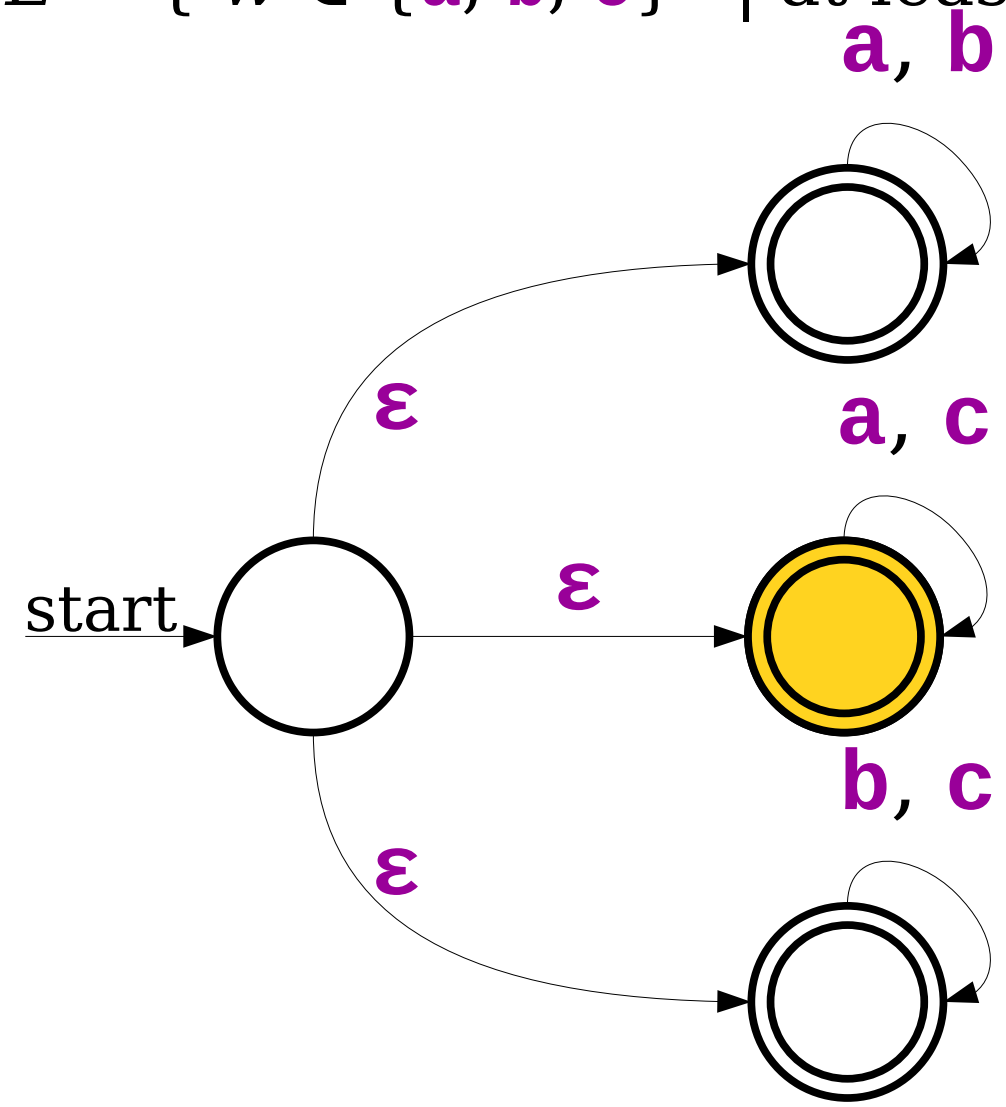

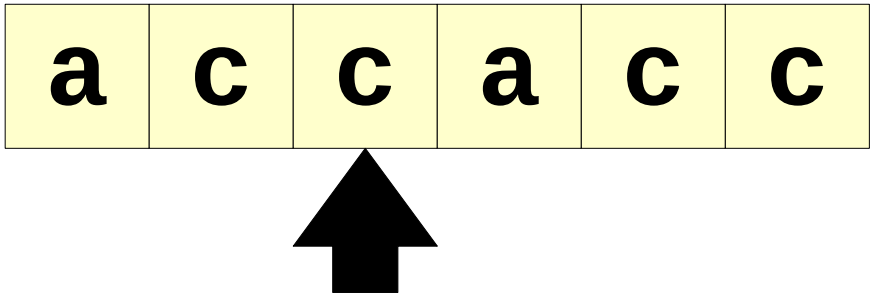

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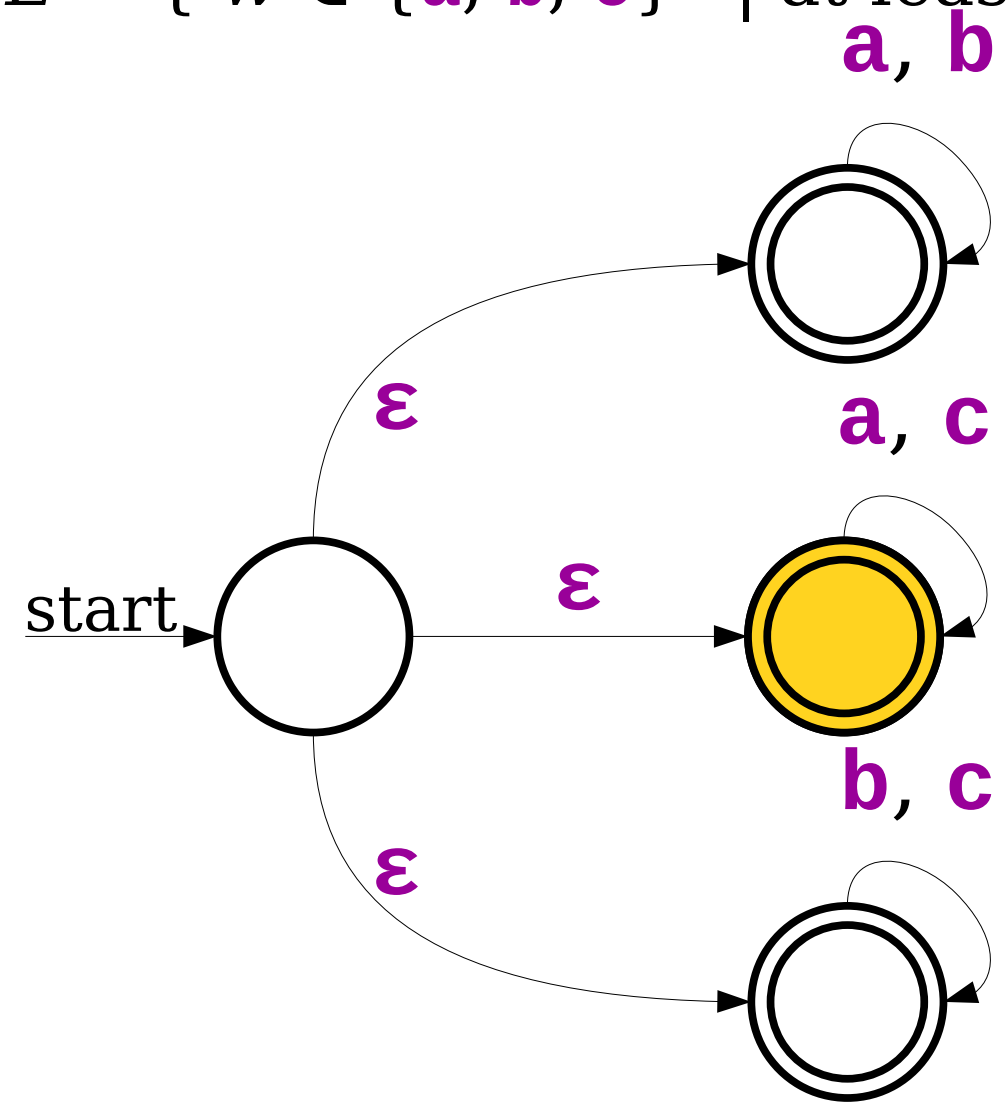

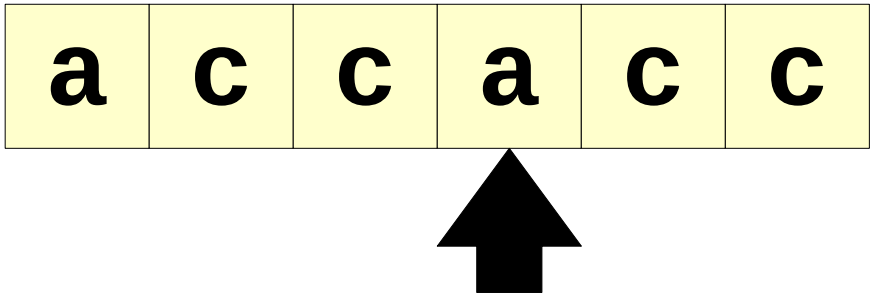

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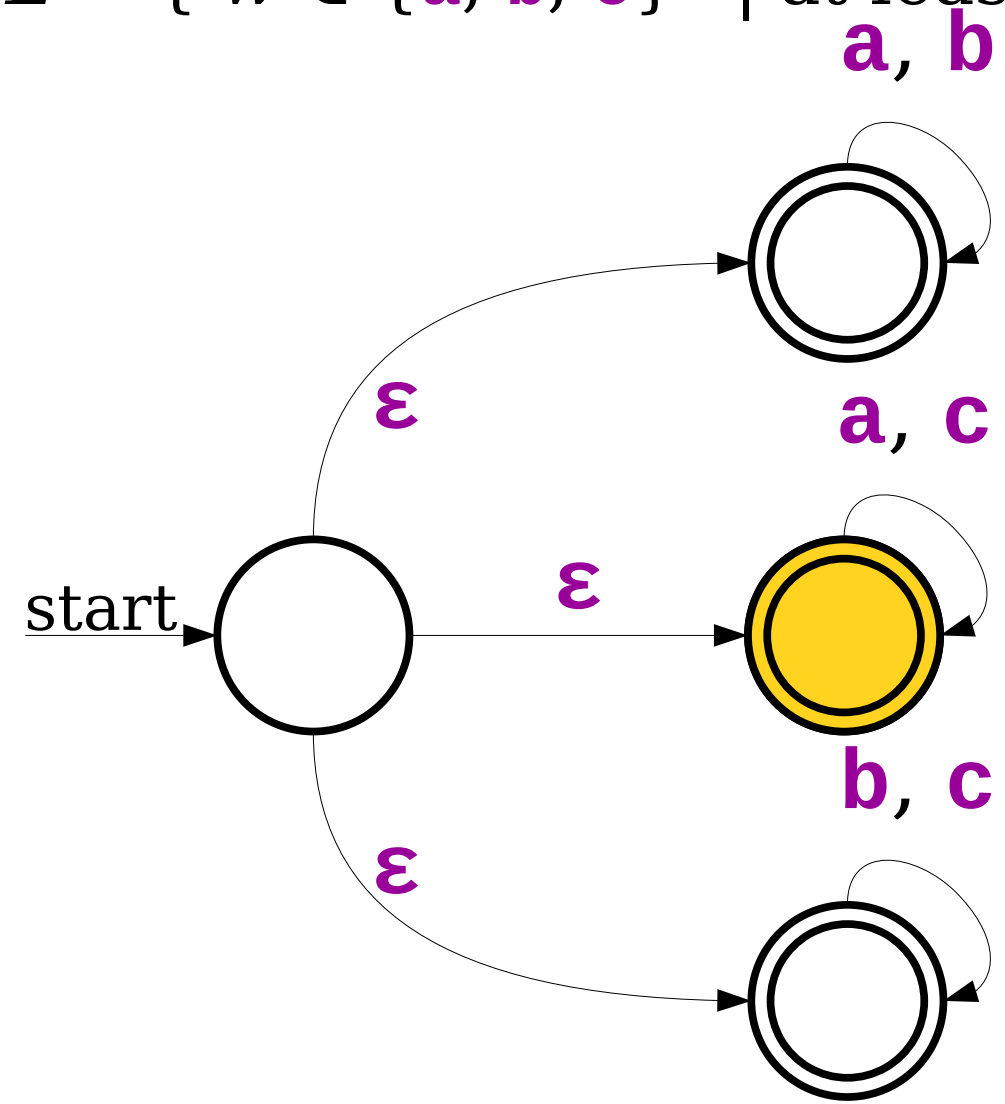

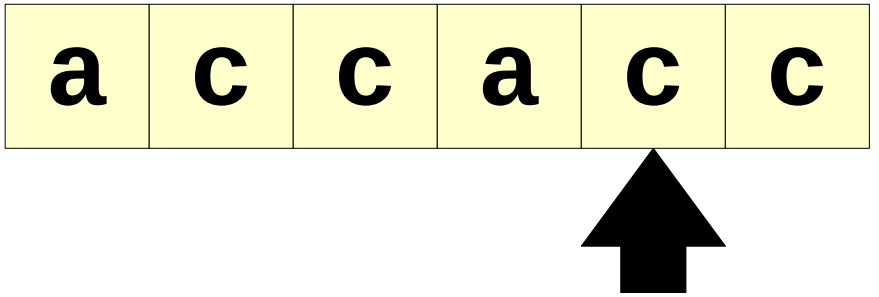

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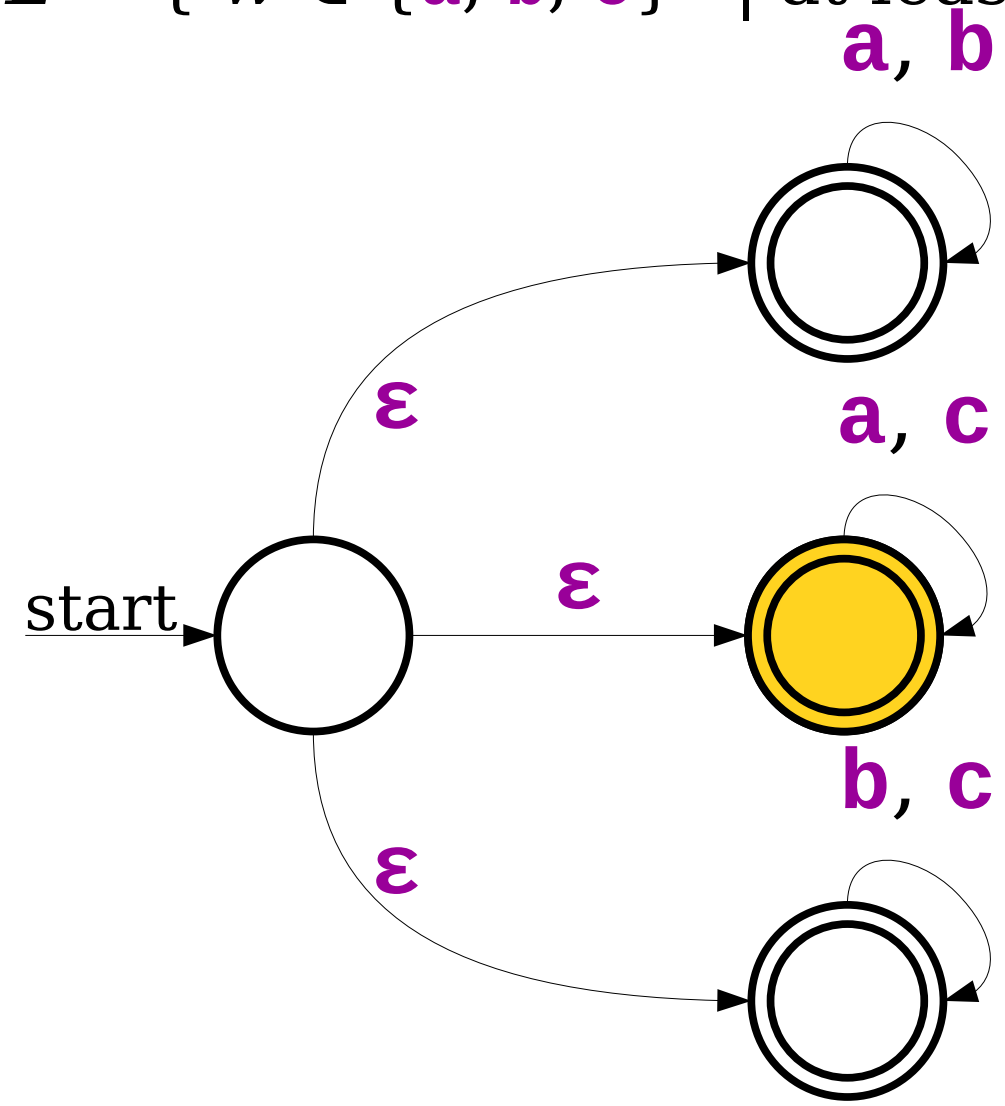

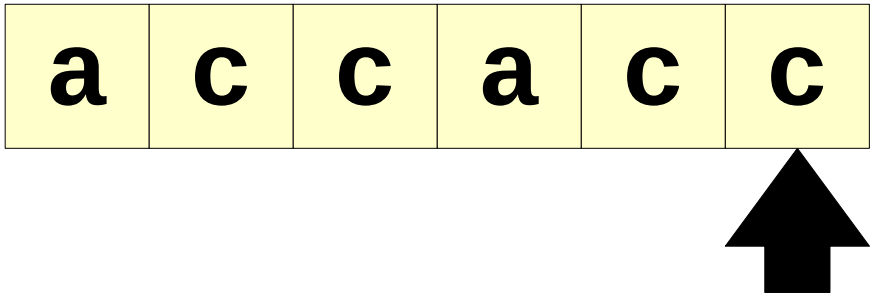

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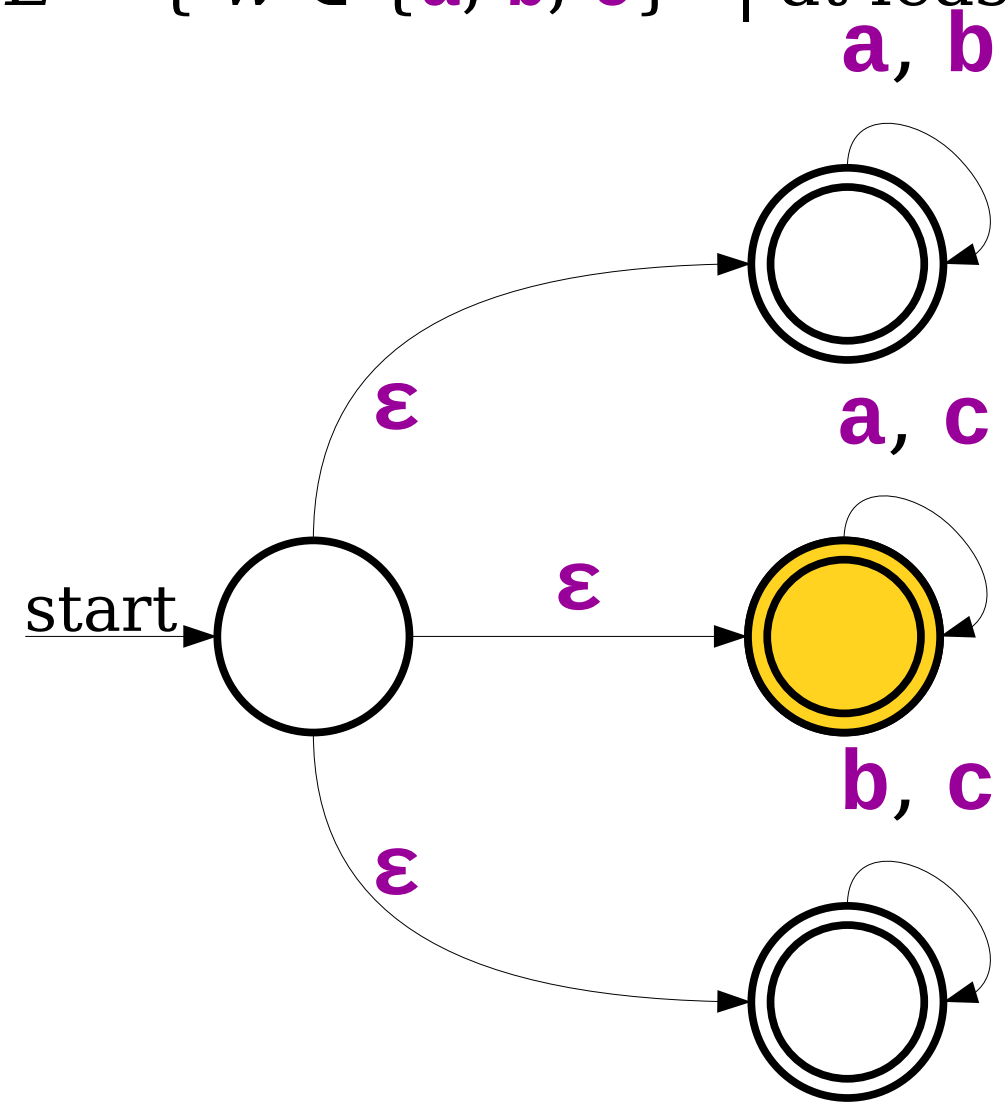


a	c	c	a	c	c

Time-Out For Announcements!

Midterm Exam on Friday!

- Our midterm exam will be on Friday, July 26th from 5:00 – 8:00 PM in Hewlett 201 (our normal lecture room).
- You're responsible for lectures up to the end of week 3 and topics from PS1 – PS3. Later lectures and problem sets won't be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is open-book, open-note, and closed-other-humans/AI.

Back to CS103!

Just how powerful are NFAs?

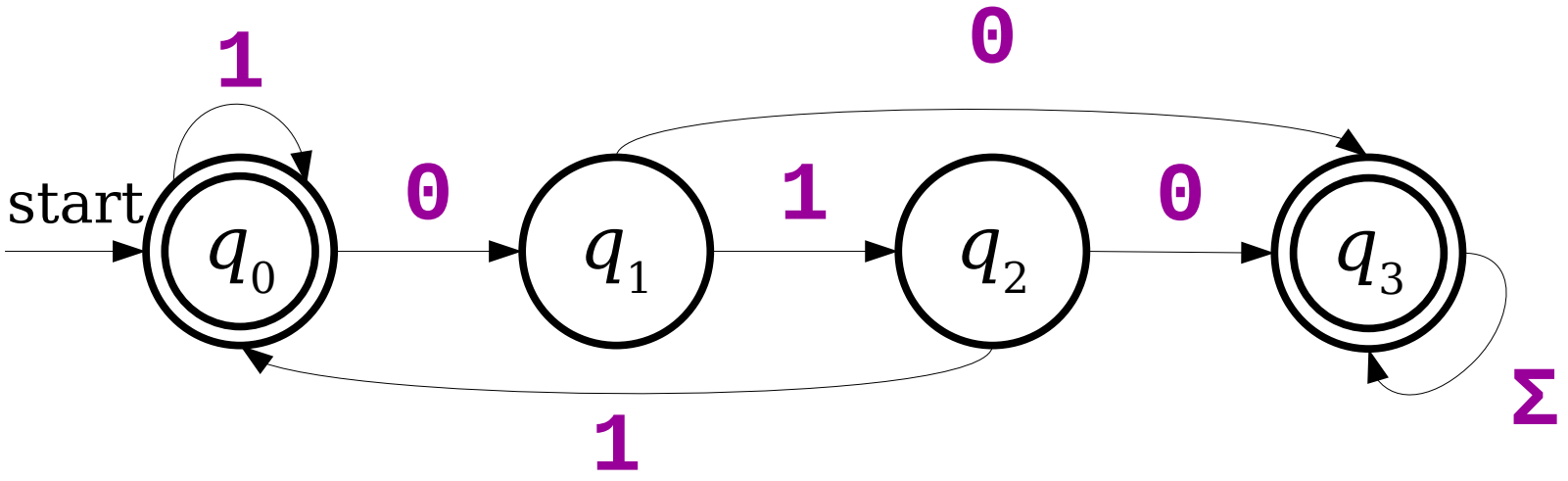
NFAs and DFAs

- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question:*** Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is ***yes***!

Thought Experiment:

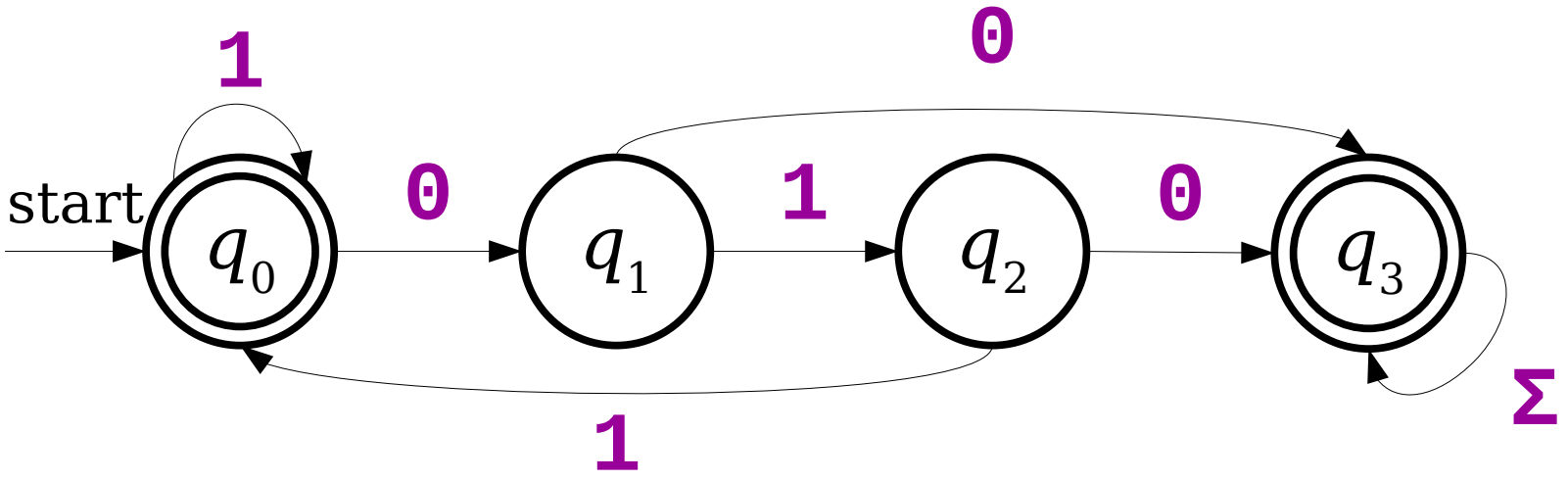
How would you simulate a finite automata in software?

Tabular DFAs



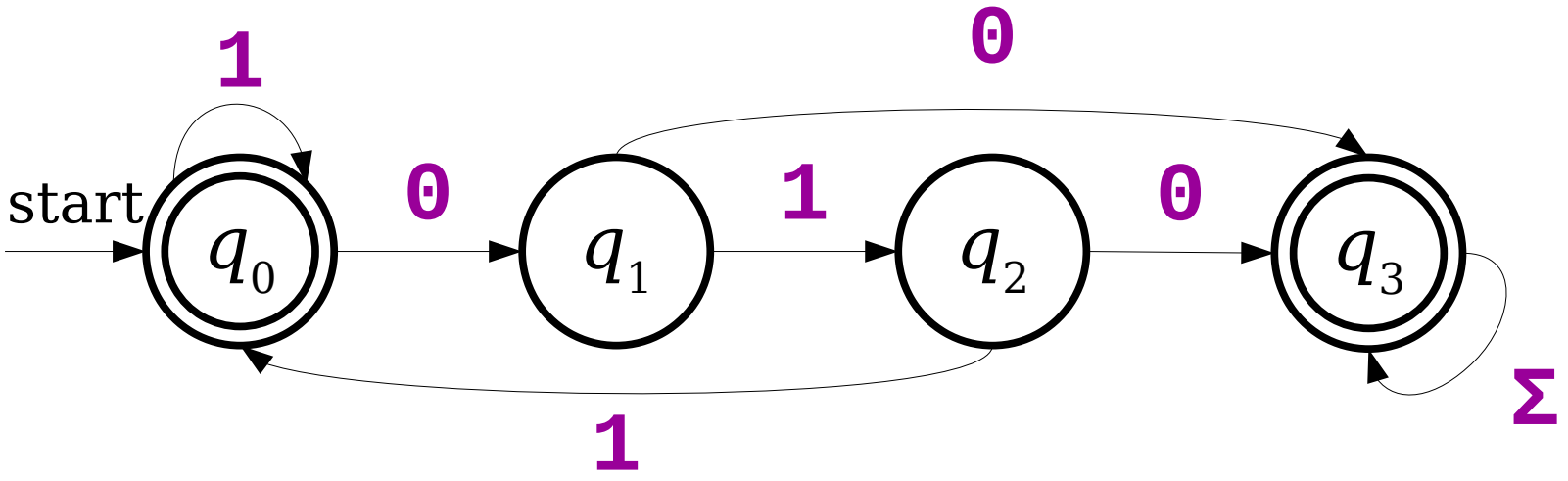
	0	1
q_0		
q_1		
q_2		
q_3		

Tabular DFAs



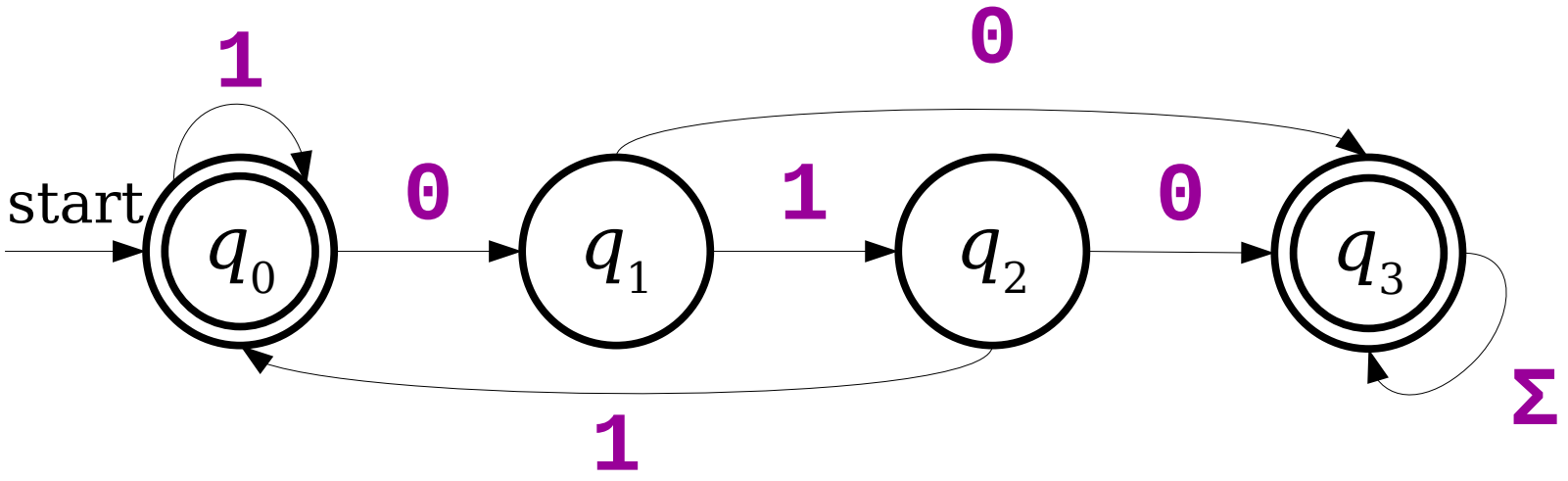
	0	1
q_0	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
q_3	q_3	q_3

Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

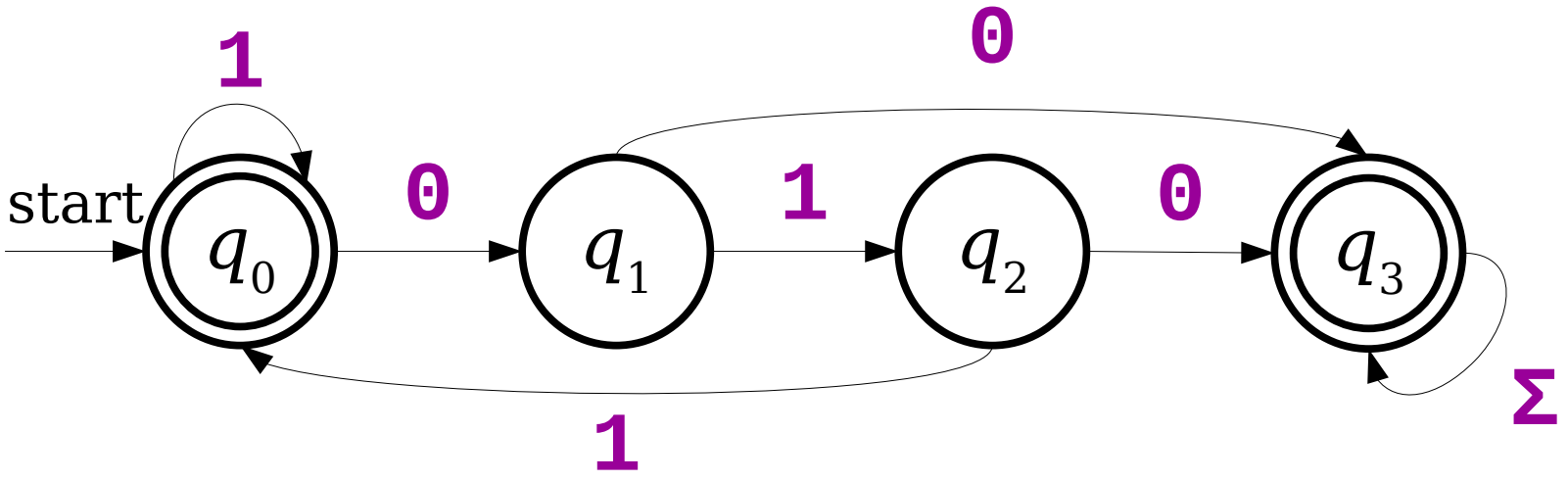
Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

These stars indicate accepting states.

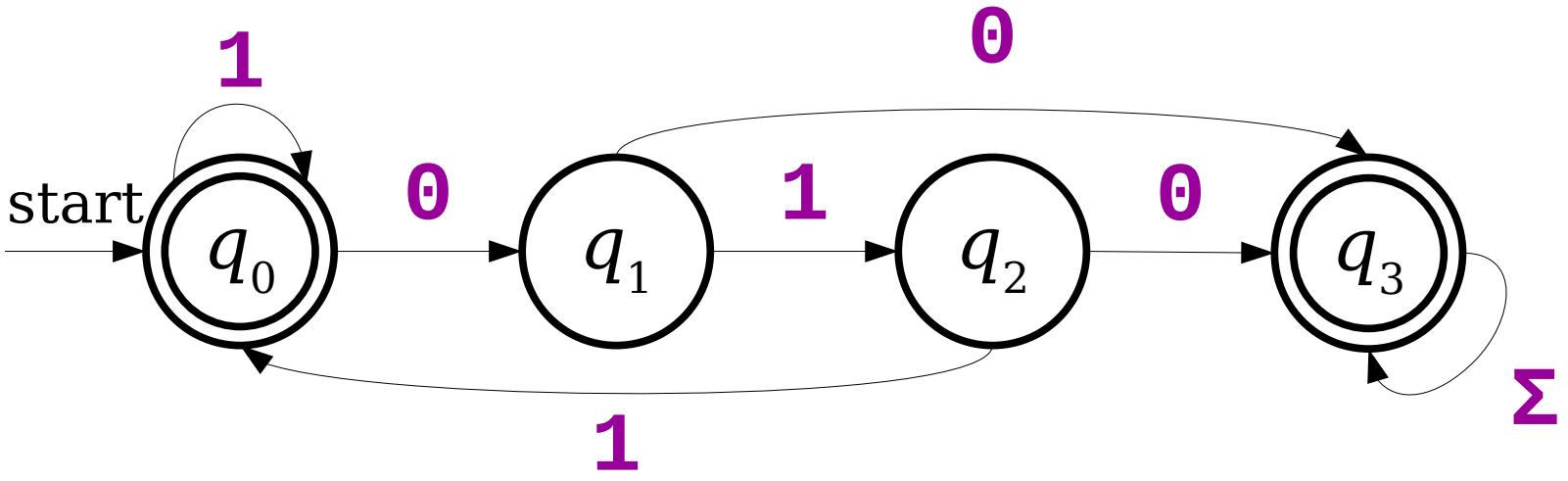
Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

Since this is the first row, it's the start state.

Tabular DFAs



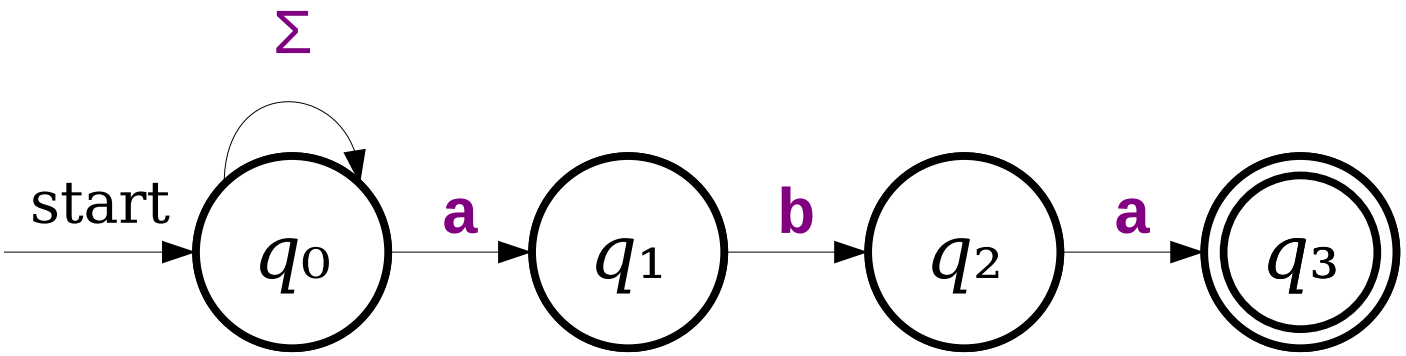
	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

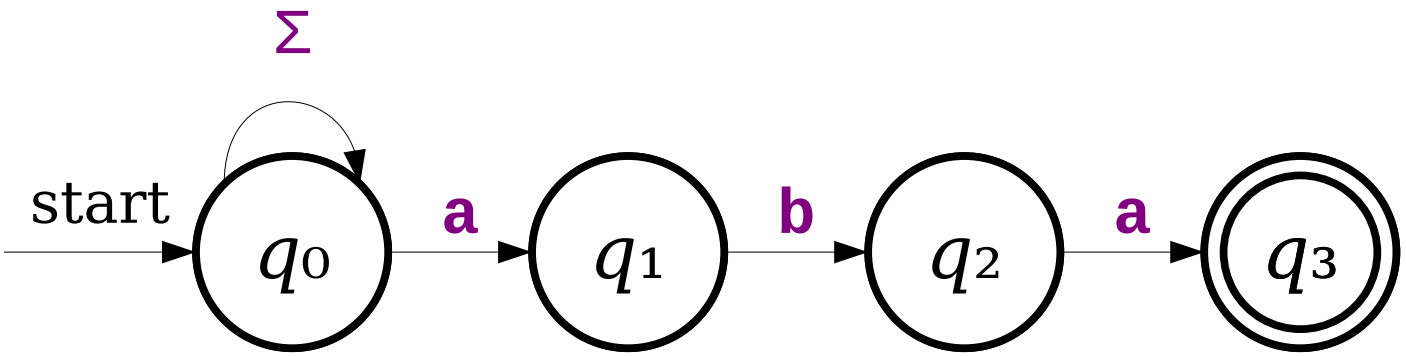
Question to ponder:
Why isn't there a column here for Σ?

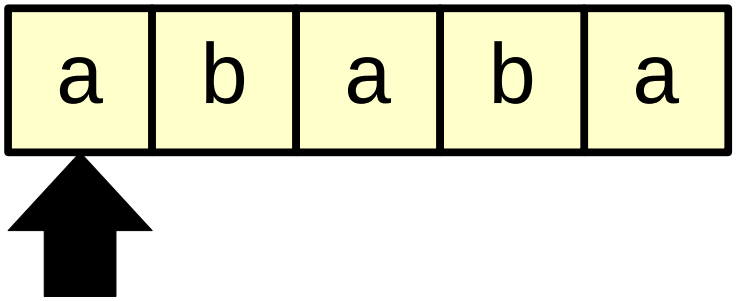
Code? In a Theory Class?

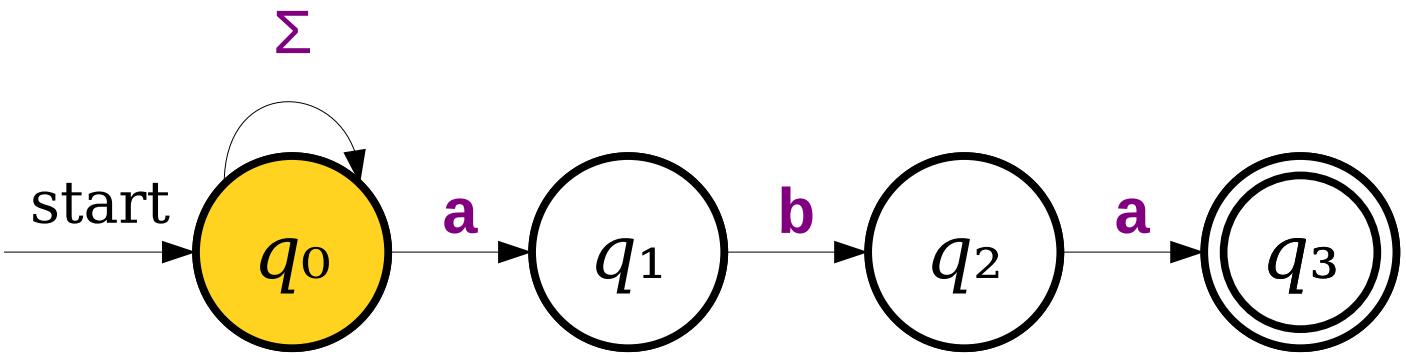
```
int kTransitionTable[kNumStates][kNumSymbols] = {
      {0, 0, 1, 3, 7, 1, …},
       …
};
bool kAcceptTable[kNumStates] = {
     false,
     true,
     true,
     …
};
bool SimulateDFA(string input) {
     int state = 0;
     for (char ch: input) {
          state = kTransitionTable[state][ch];
     }
     return kAcceptTable[state];
}
```

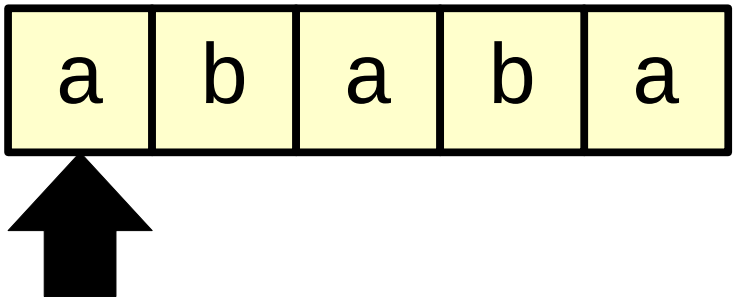
Can we do something similar for NFAs?

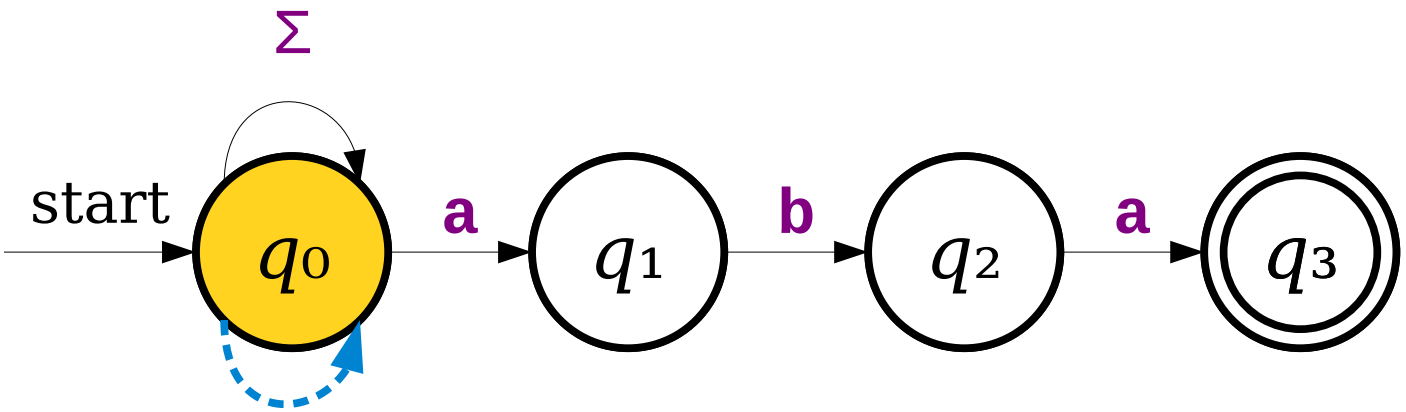

Σ
start
q_0
a
q_1
b
q_2
a
q_3

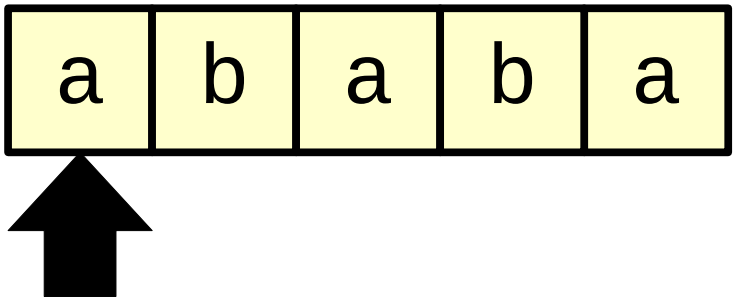

Σ
start
q_0
a
q_1
b
q_2
a
q_3

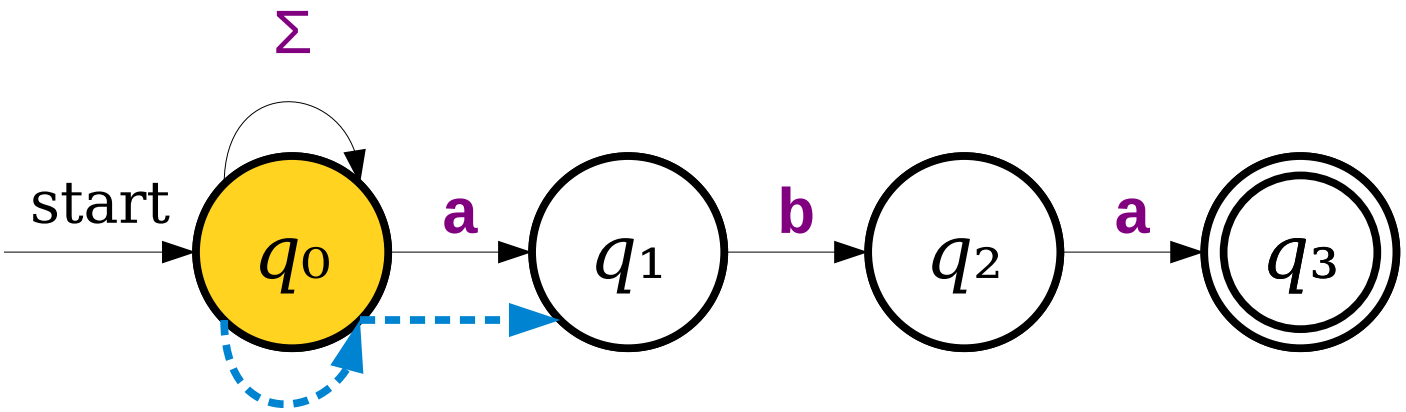

a
b
a
b
a

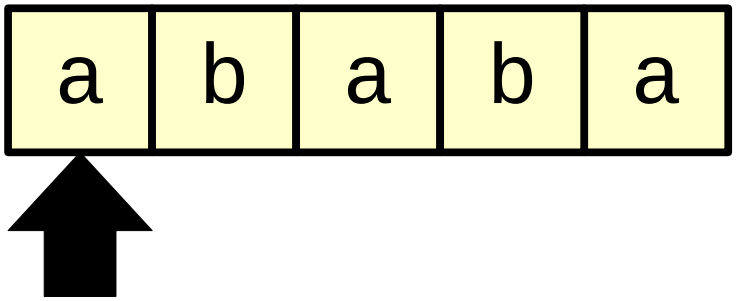

Σ
start
q_0
a
q_1
b
q_2
a
q_3

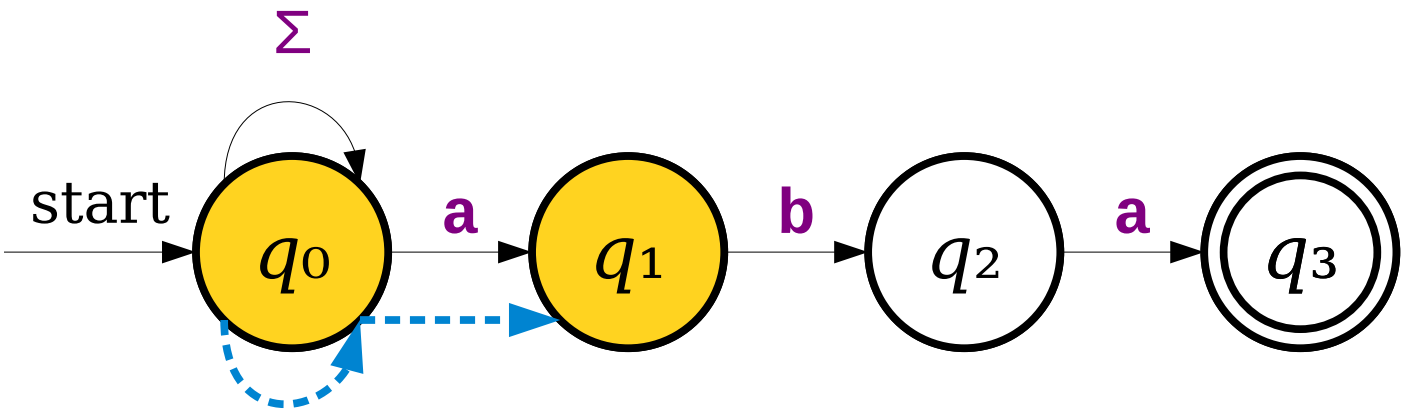

a
b
a
b
a

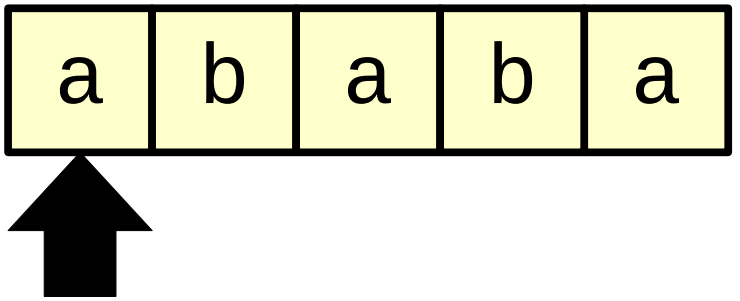

Σ
start
q_0
a
q_1
b
q_2
a
q_3

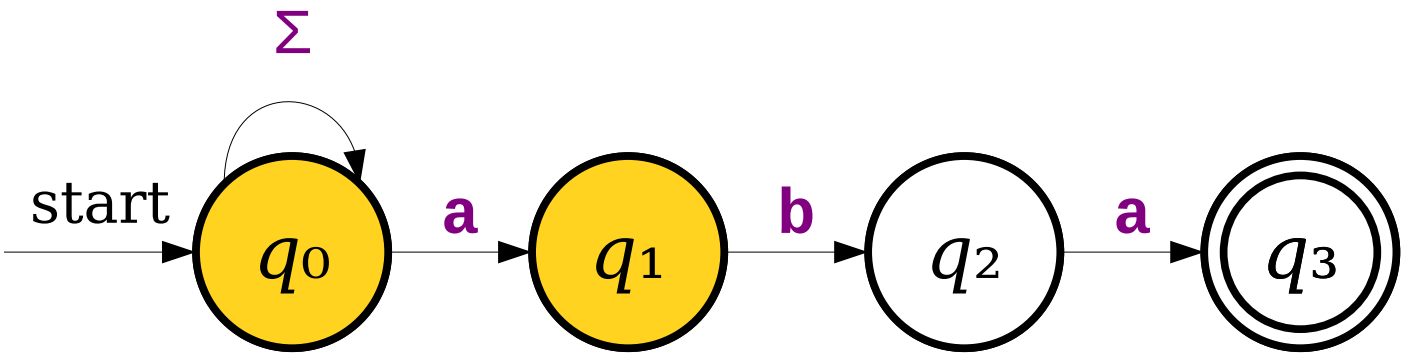

a
b
a
b
a

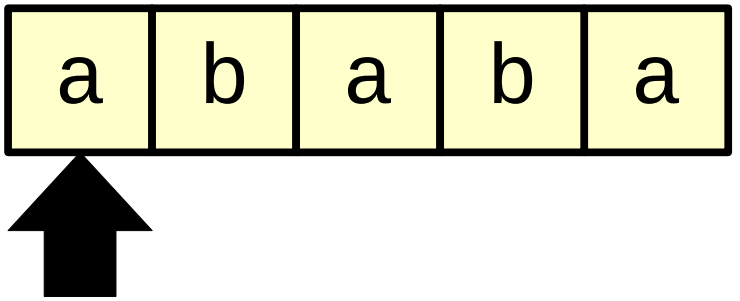

Σ
start
q_0
a
q_1
b
q_2
a
q_3

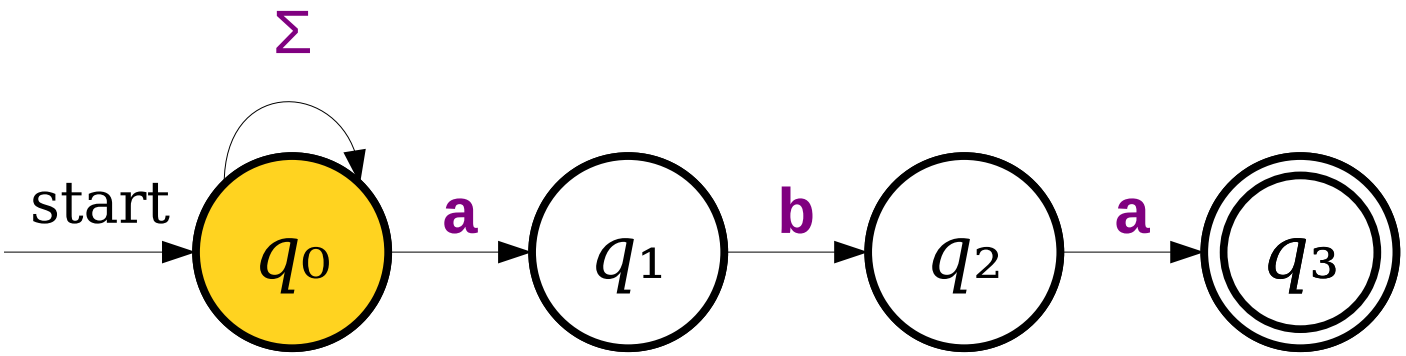

a
b
a
b
a


Σ
start
q_0
a
q_1
b
q_2
a
q_3


a
b
a
b
a


Σ
start
q_0
a
q_1
b
q_2
a
q_3


a
b
a
b
a


Σ
start
q_0
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q_1
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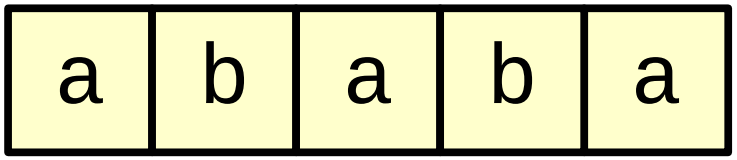
q_2
a
q_3


a
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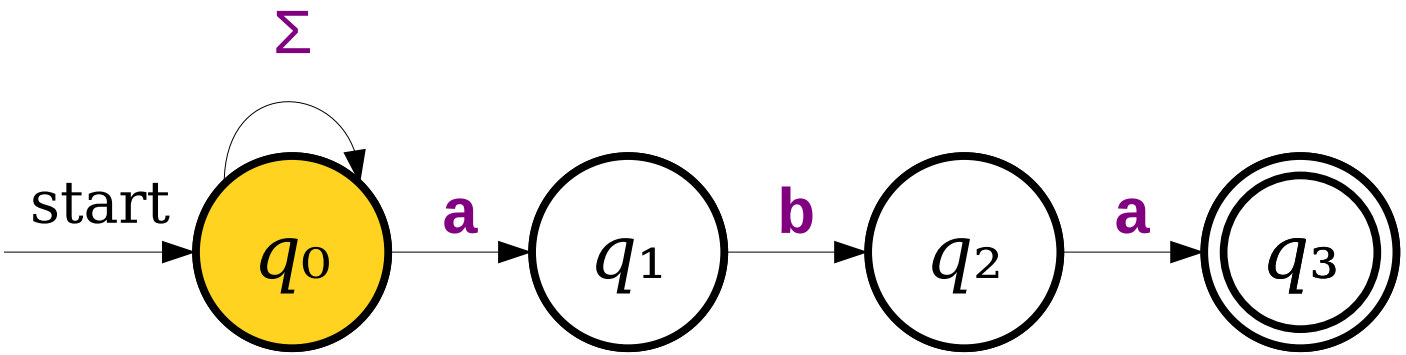
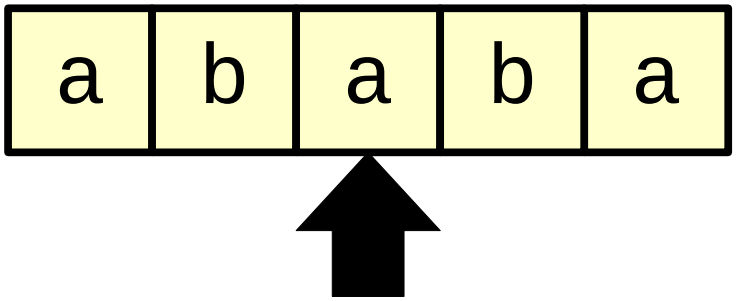
a


Σ
start
q_0
a
q_1
b
q_2
a
q_3


a
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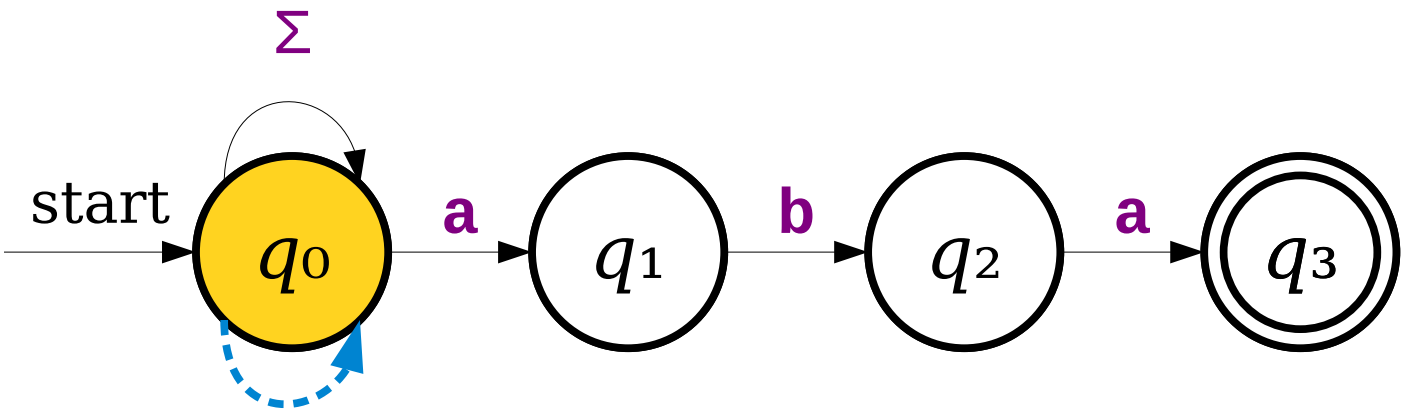
Σ
start
q_0
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q_1
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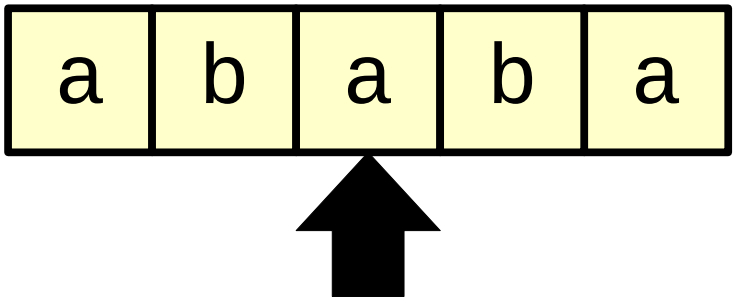
q_3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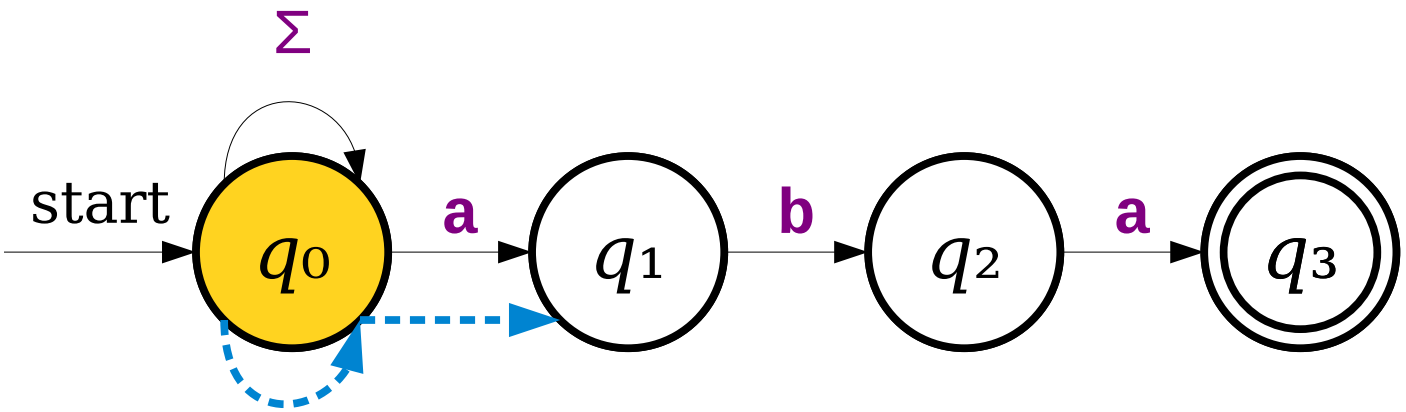
Σ
start
q_0
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q_1
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q_2
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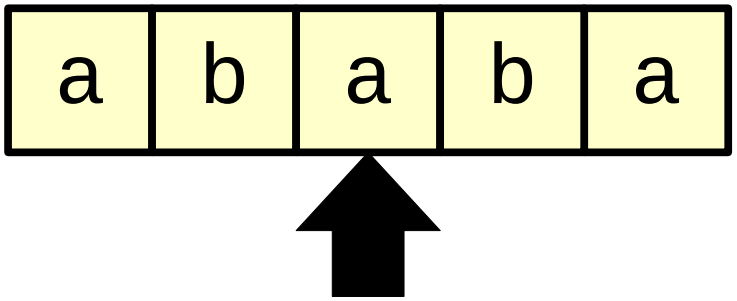
q_3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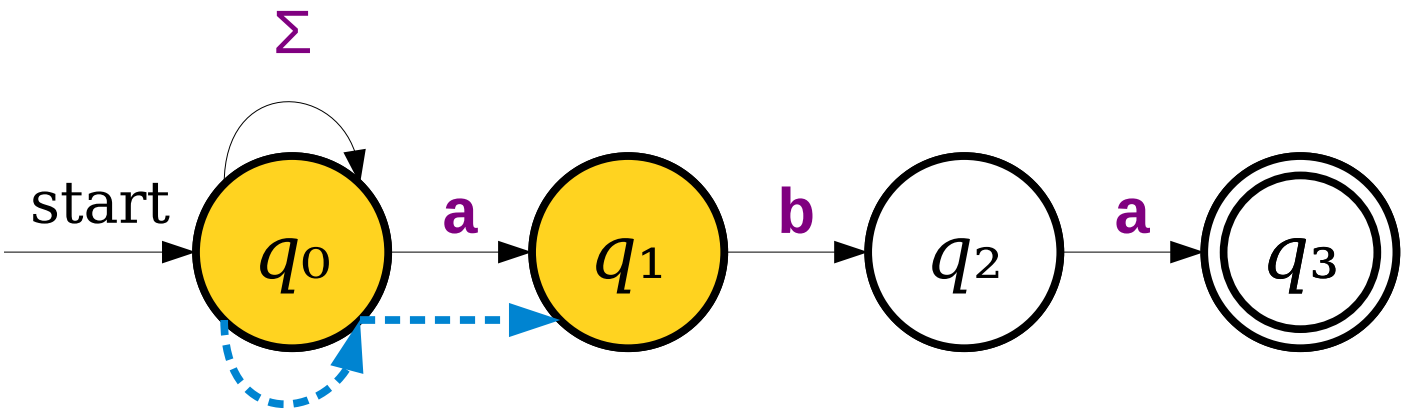
Σ
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q_0
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q_1
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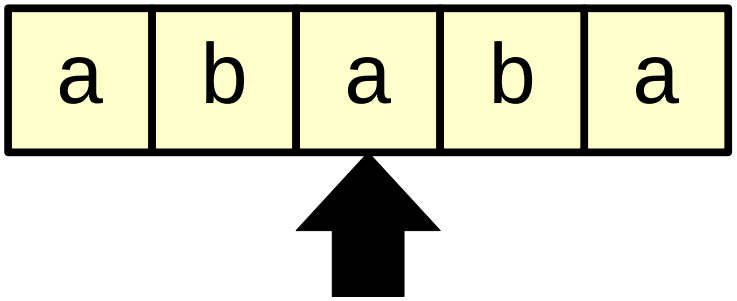
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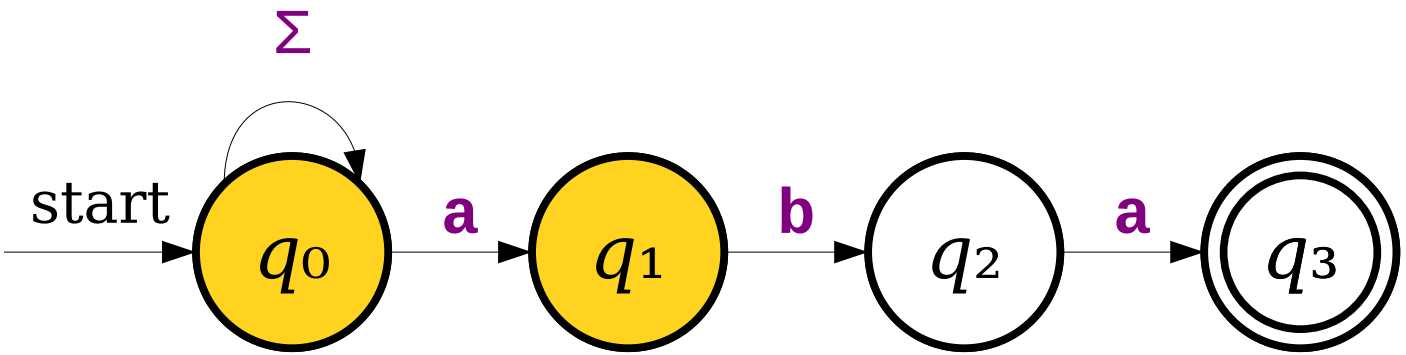
Σ
start
q_0
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q_1
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q_2
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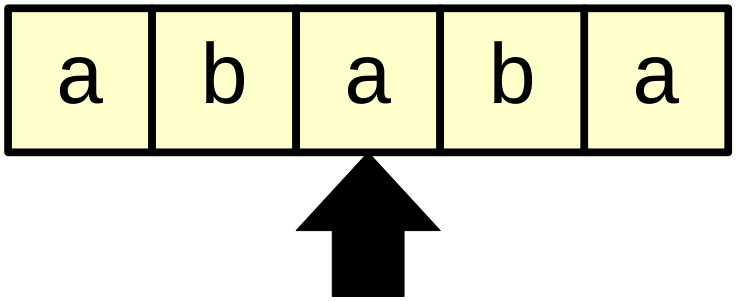
q_3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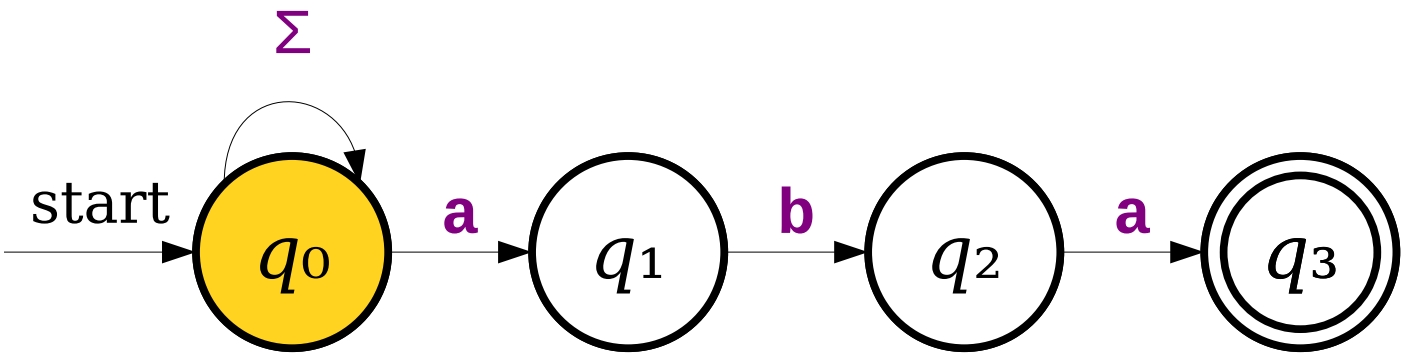
Σ
start
q_0
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q_1
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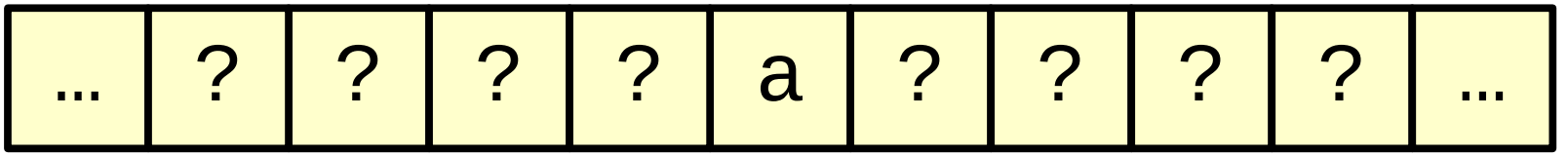
q_2
a
q_3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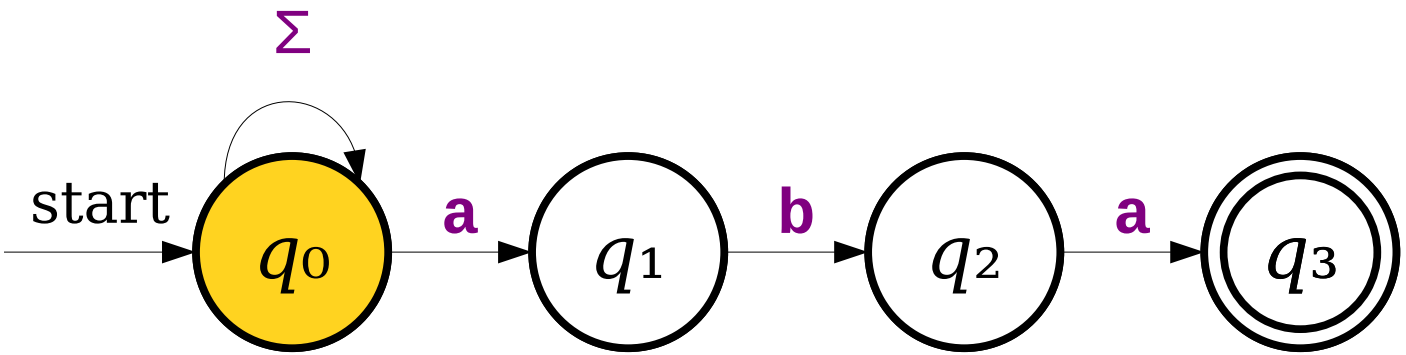
Σ
start
q_0
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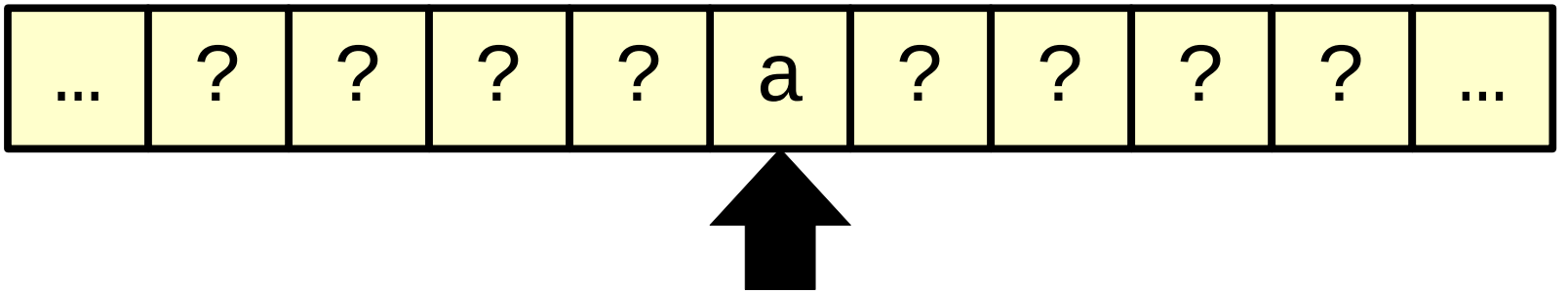
q_1
b
q_2
a
q_3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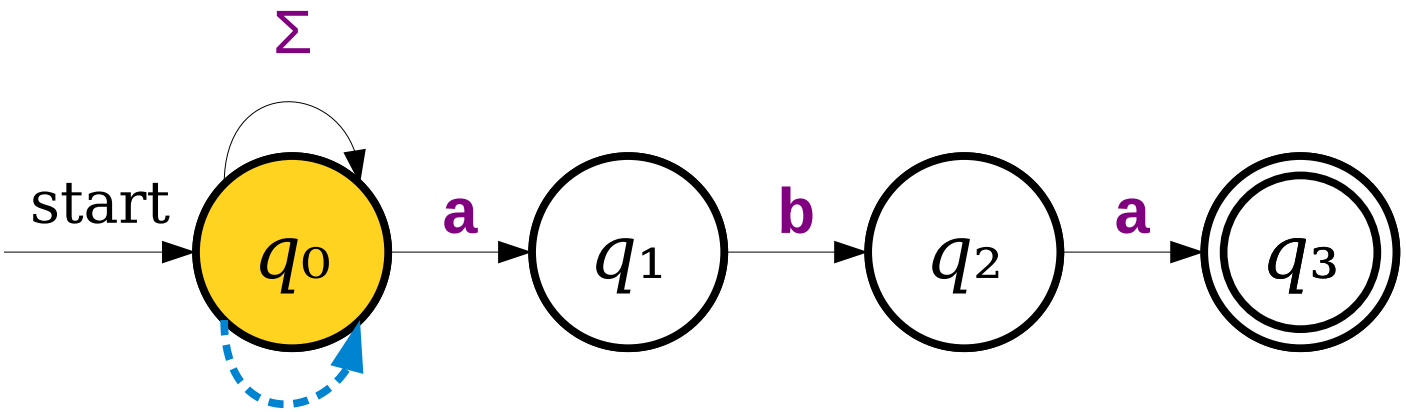
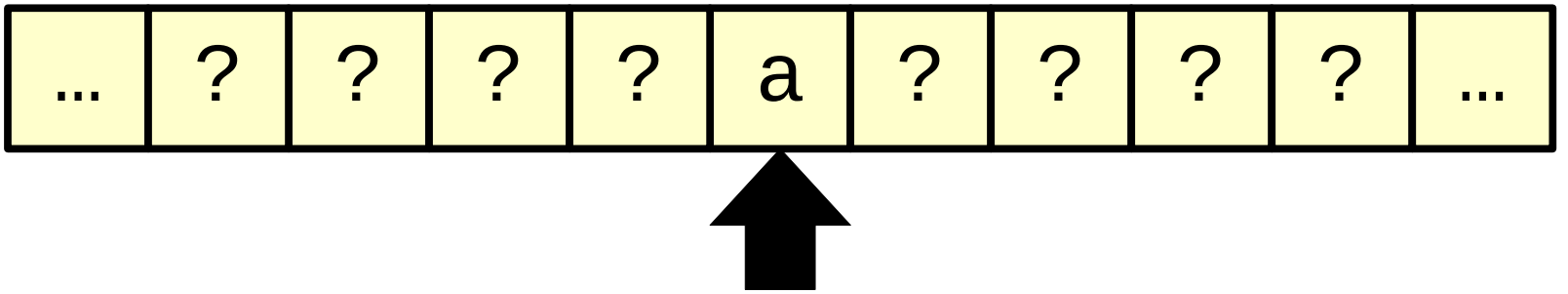

Σ
start
q_0
a
q_1
b
q_2
a
q_3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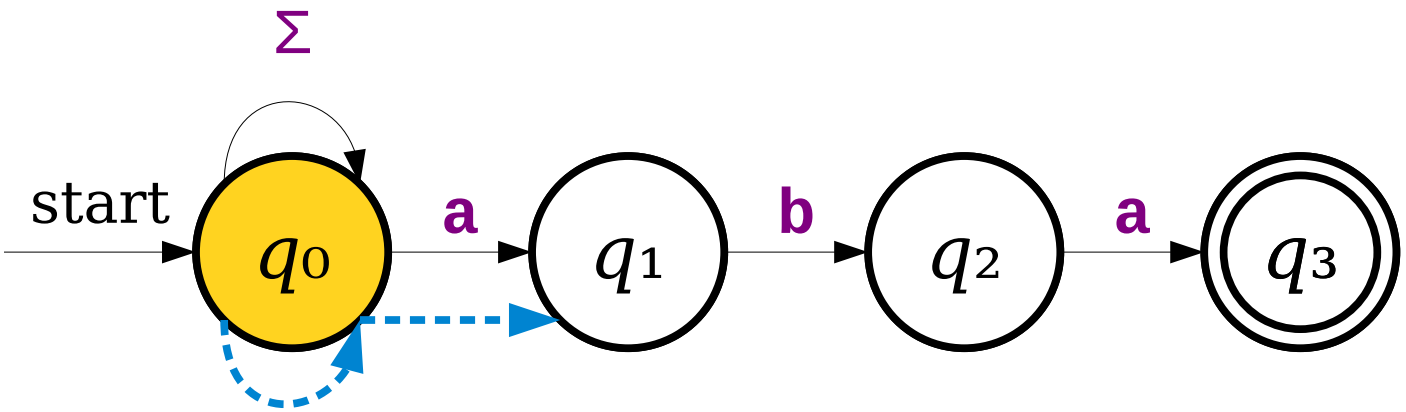
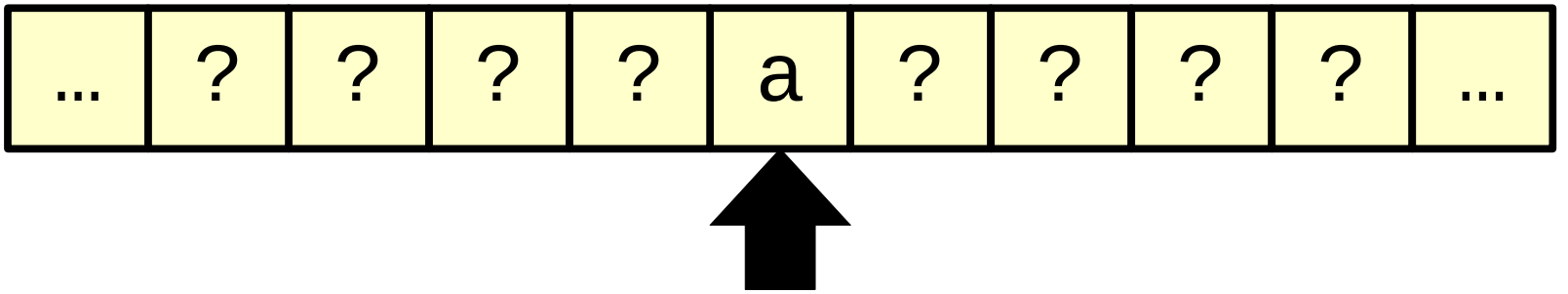

Σ
start
q_0
a
q_1
b
q_2
a
q_3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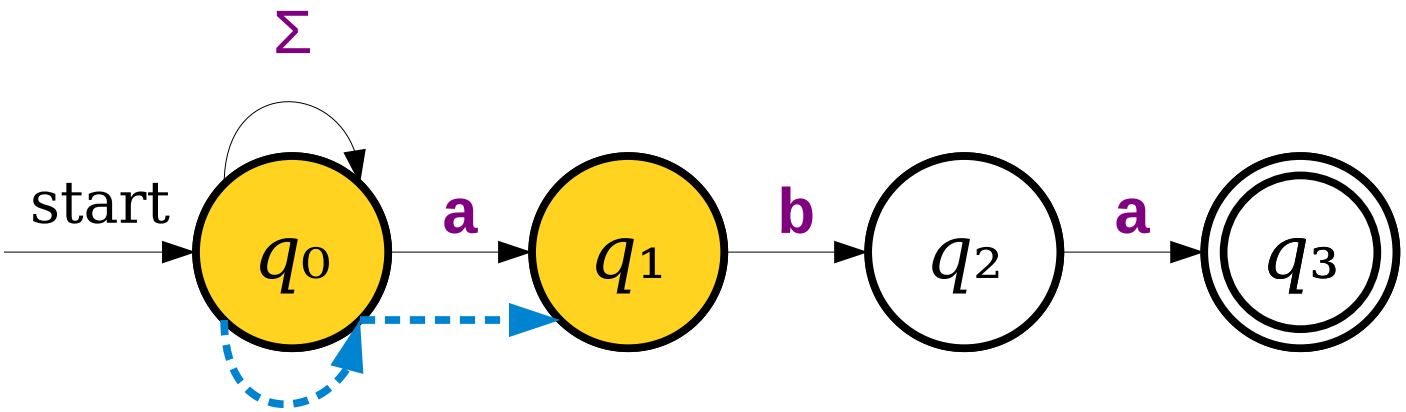
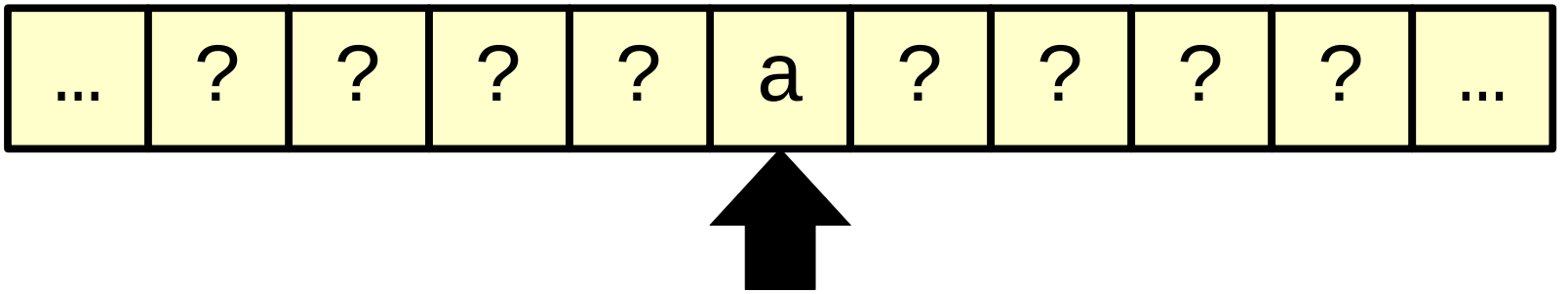

Σ
start
q_0
a
q_1
b
q_2
a
q_3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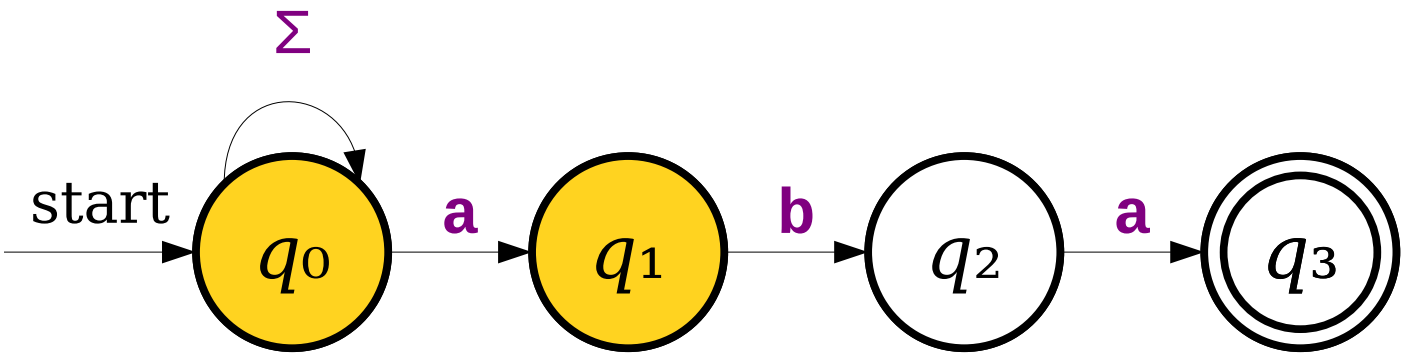
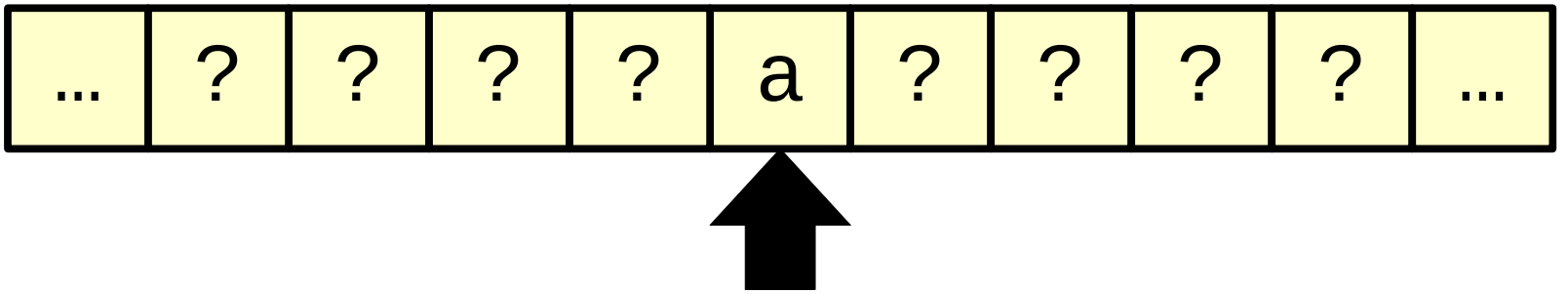

Σ
start
q_0
a
q_1
b
q_2
a
q_3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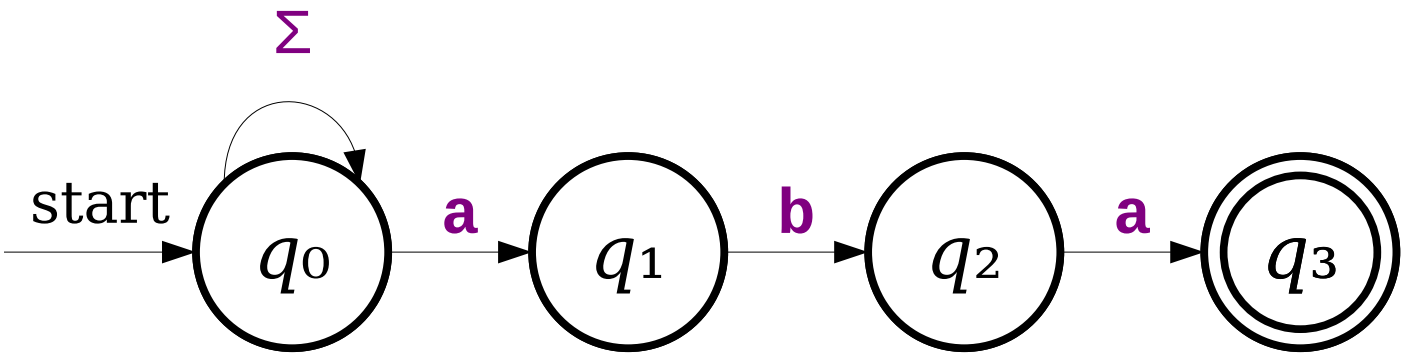
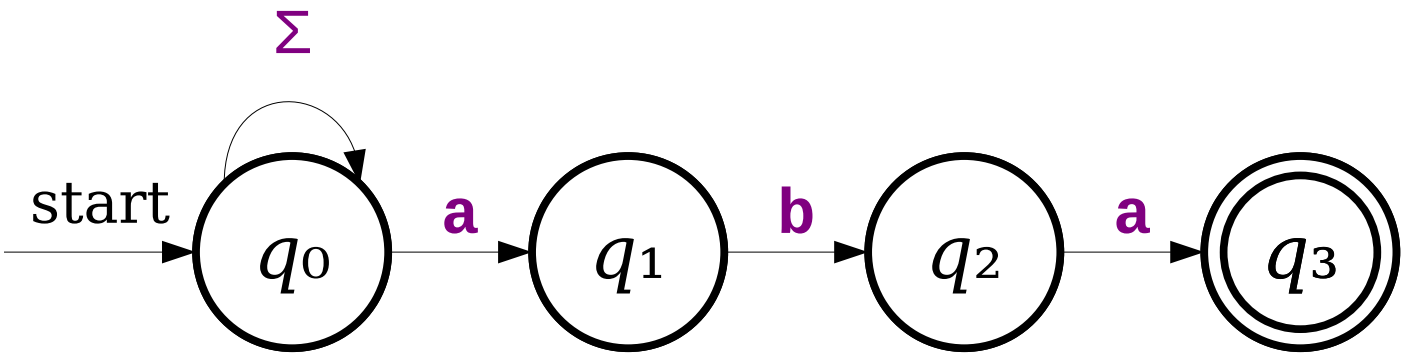
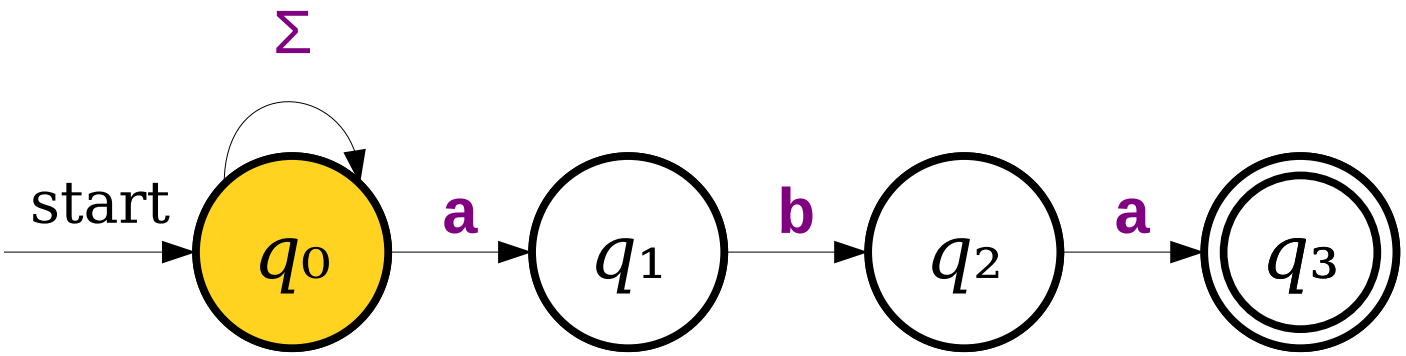


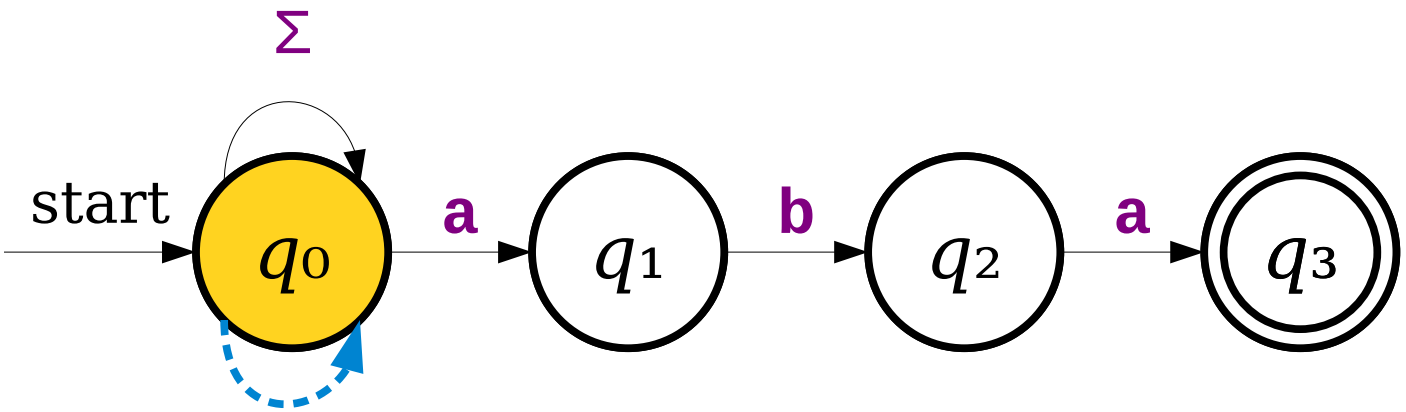
	a
$\{q_0\}$	$\{q_0, q_1\}$



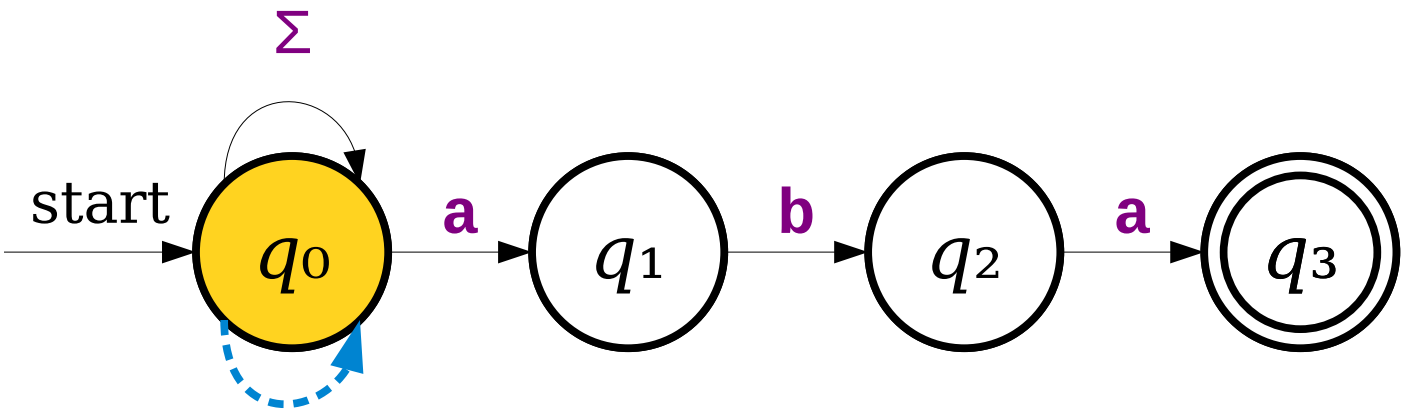
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	



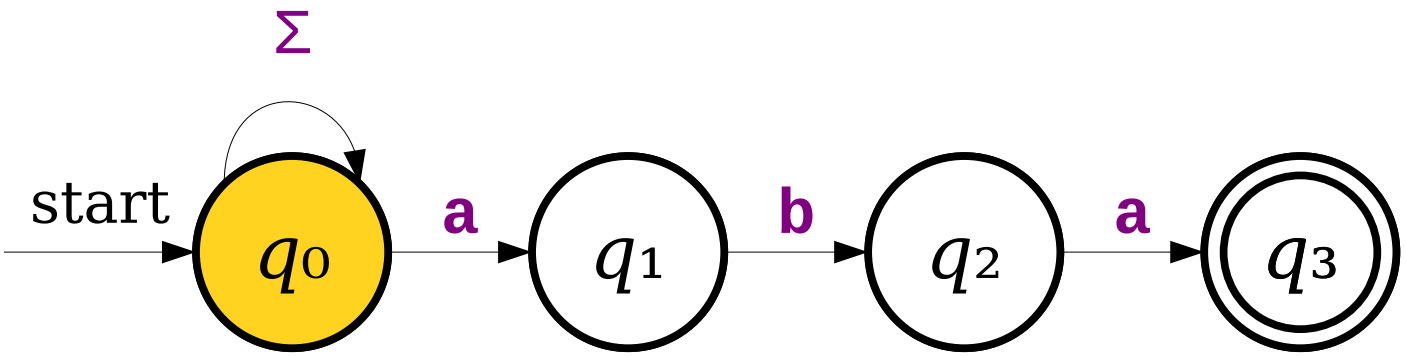
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	



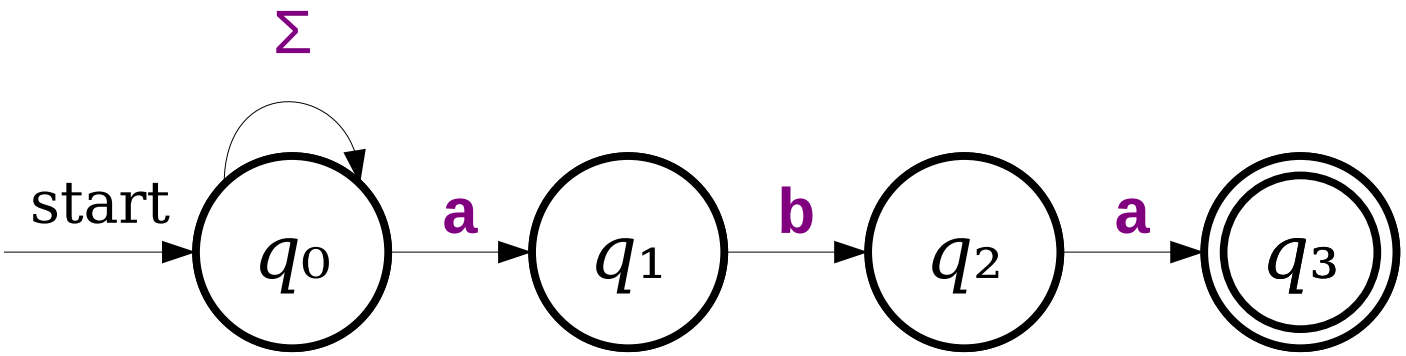
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$



	a	b
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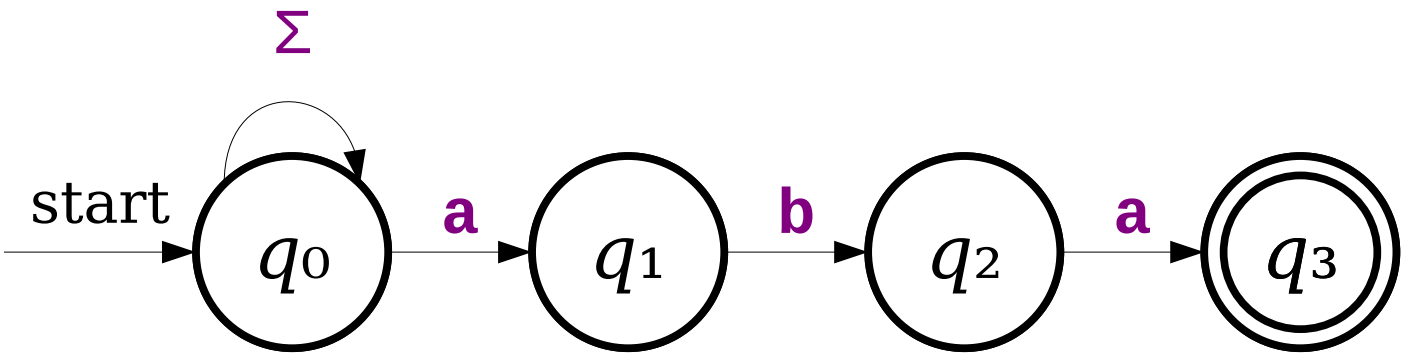

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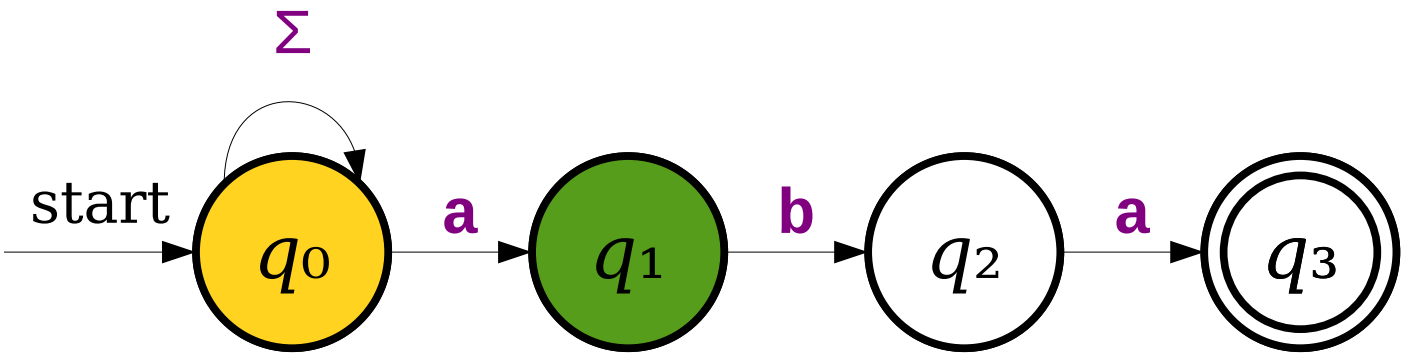

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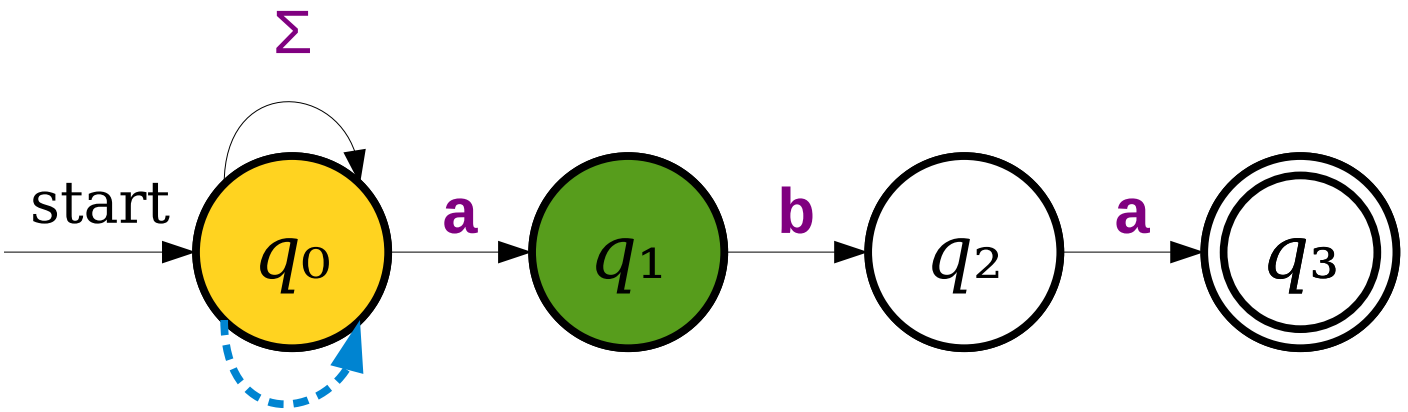

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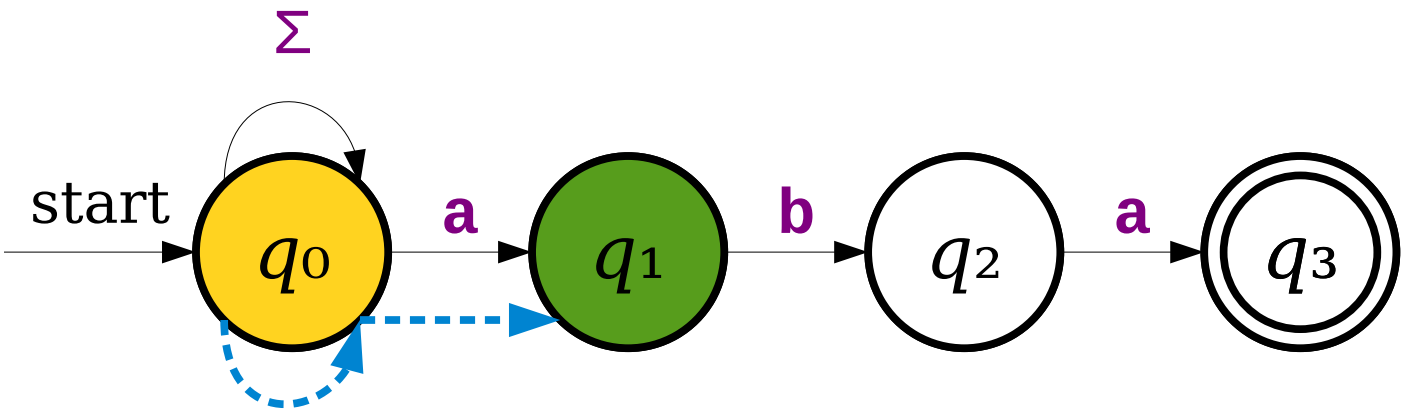

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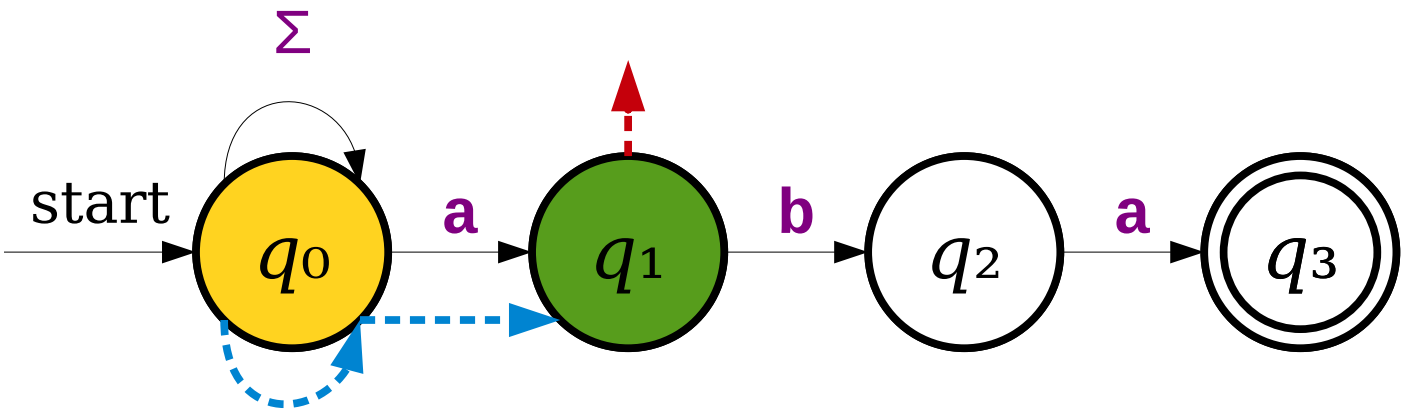

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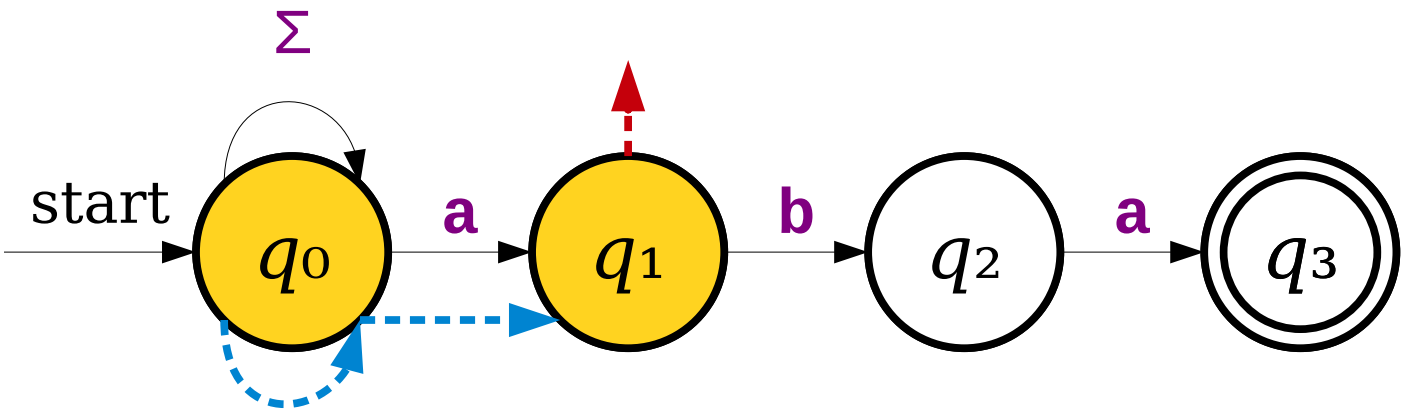

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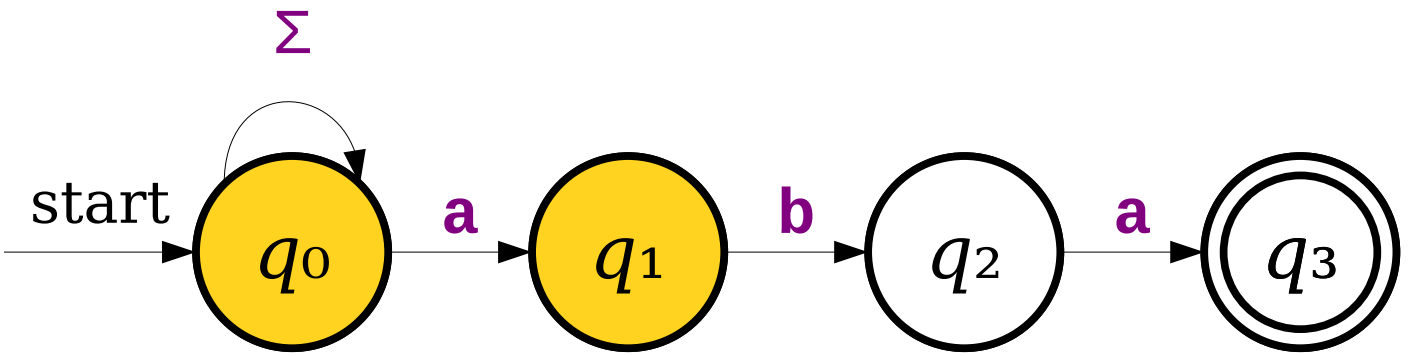

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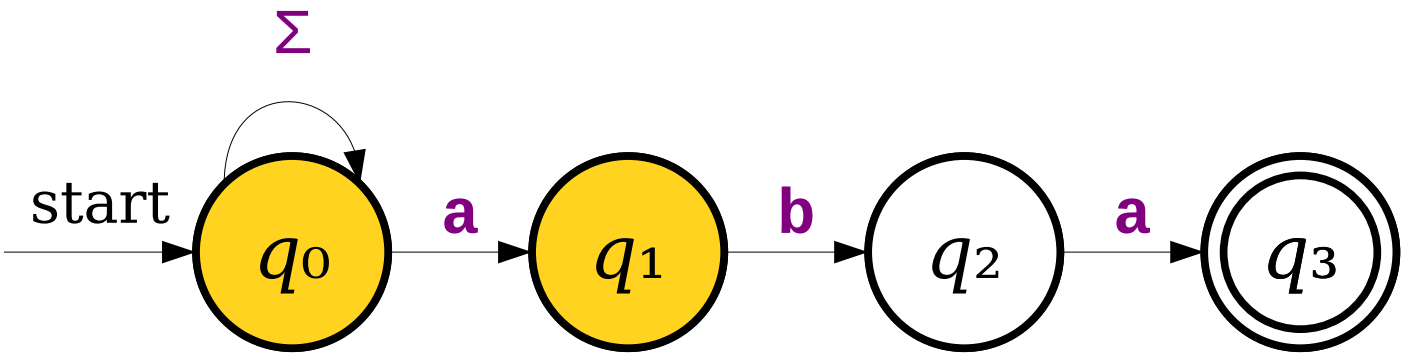

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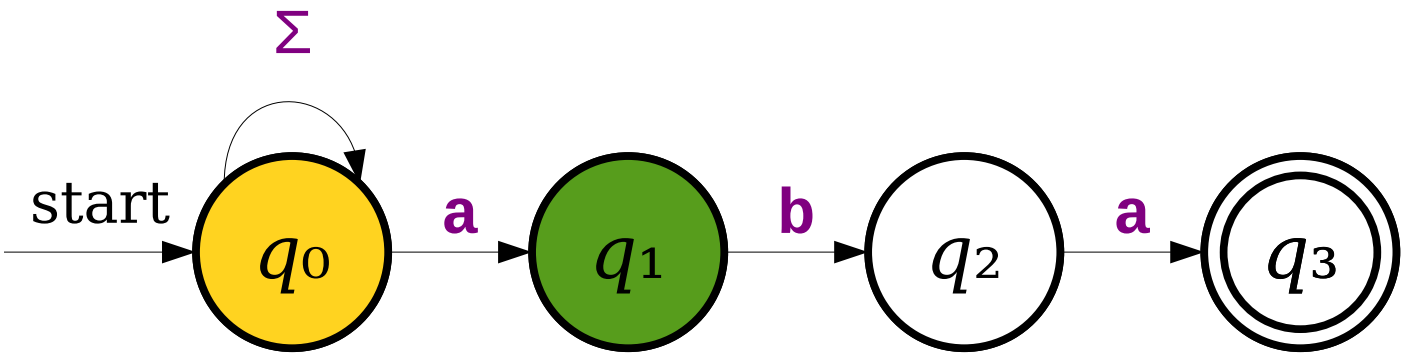

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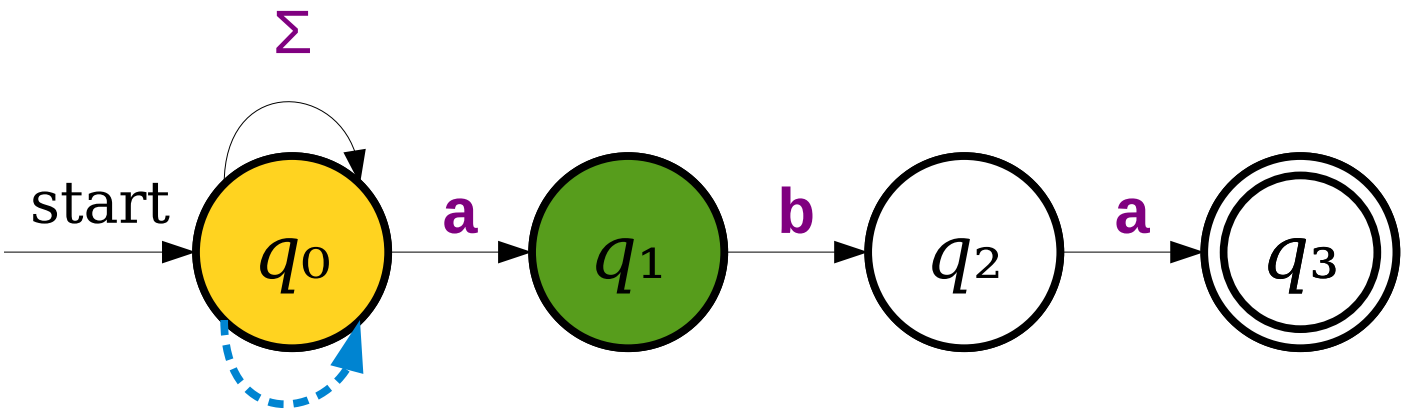

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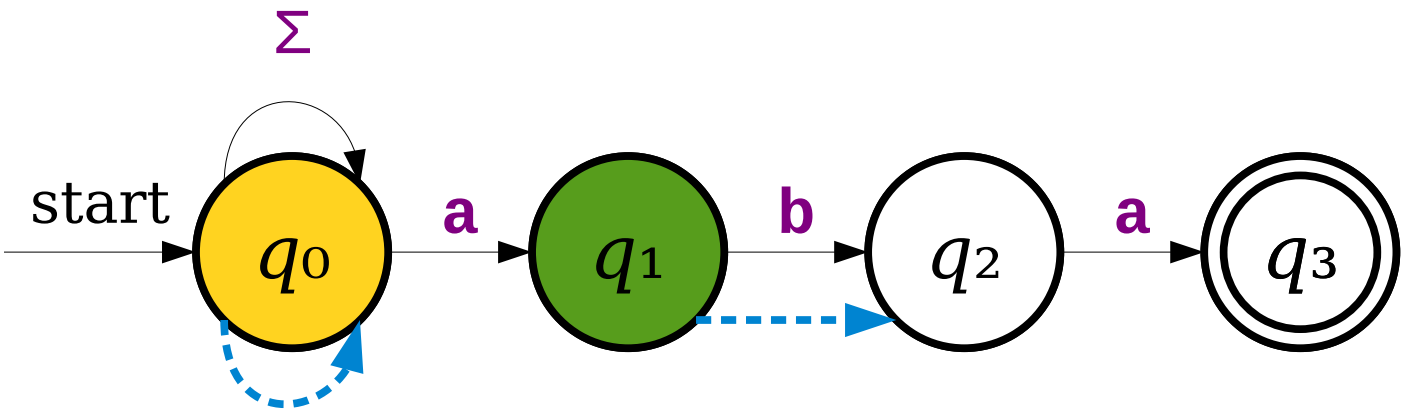

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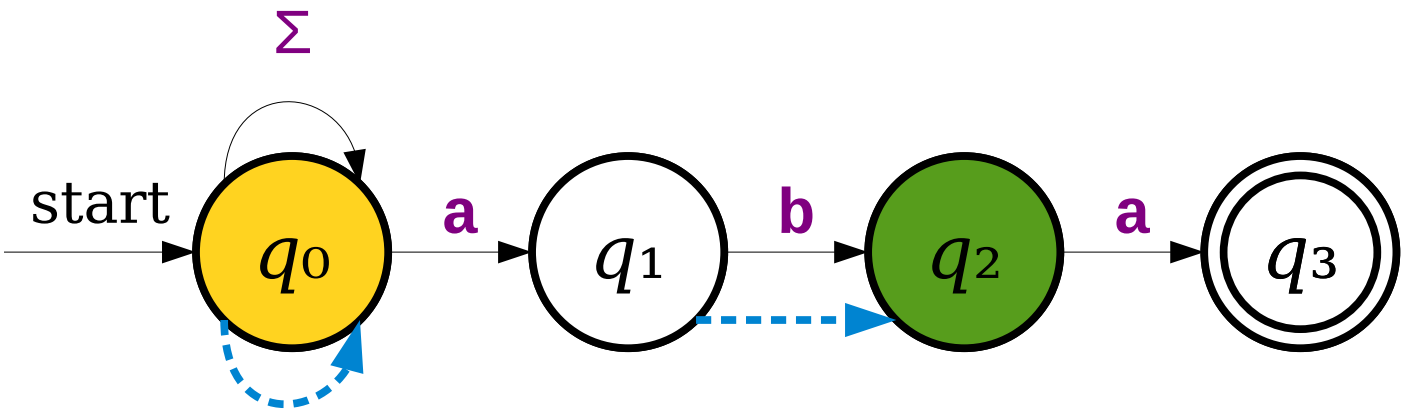

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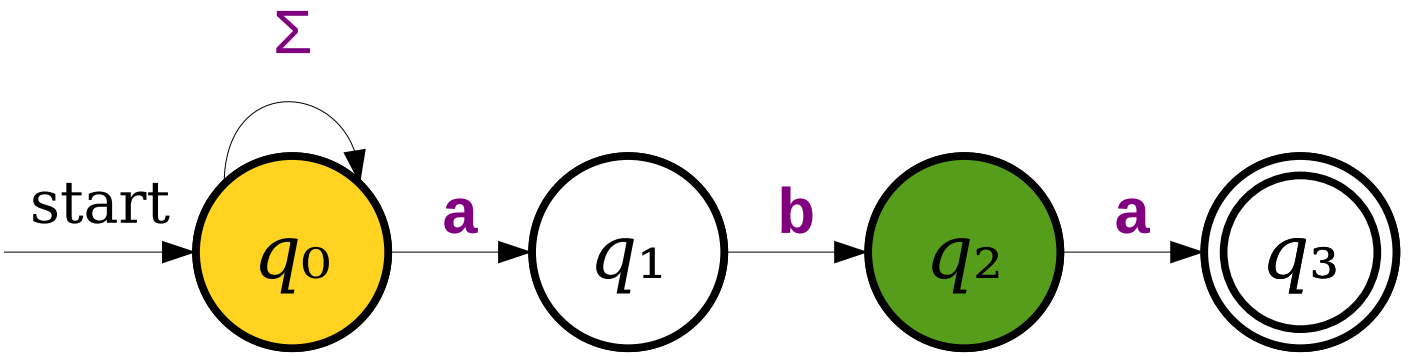

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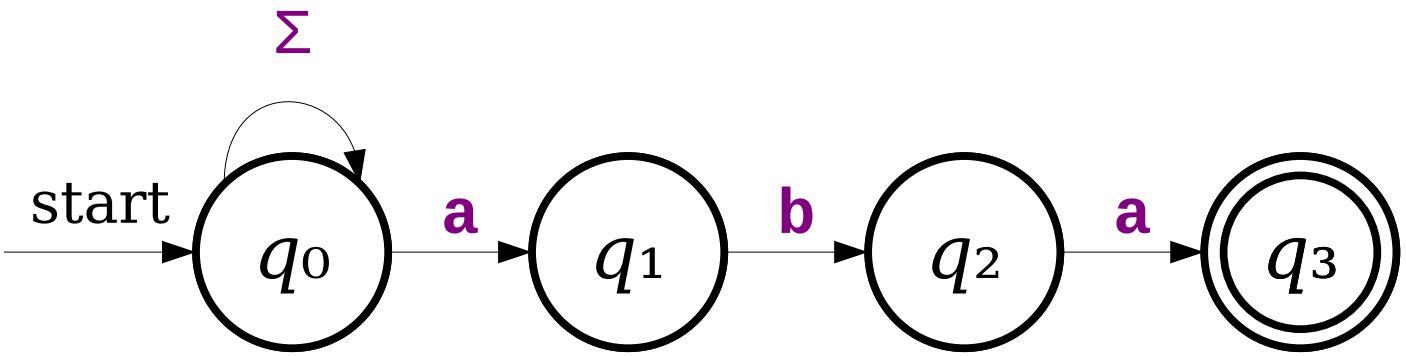

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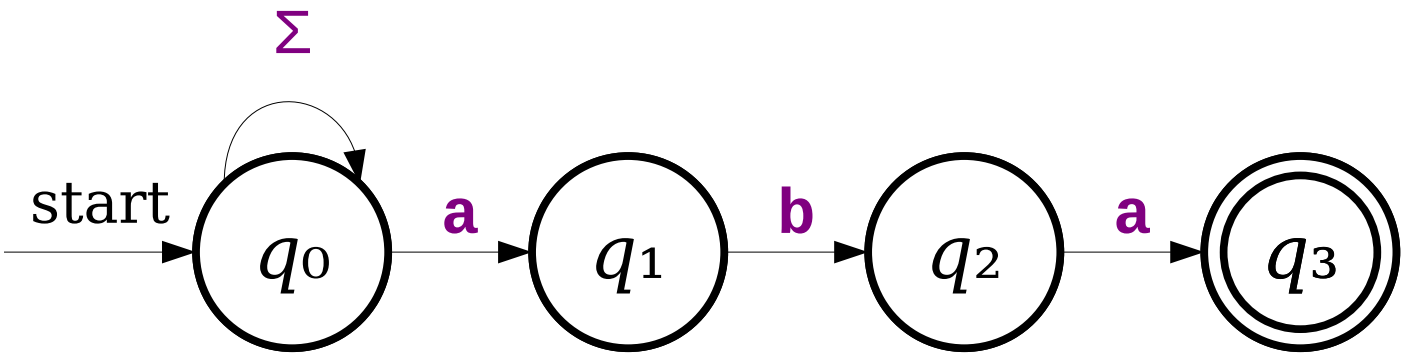

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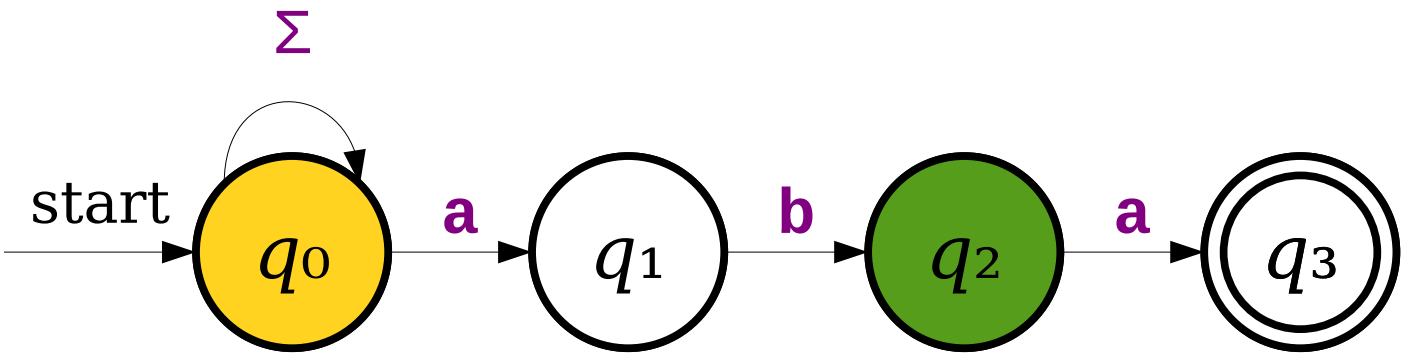

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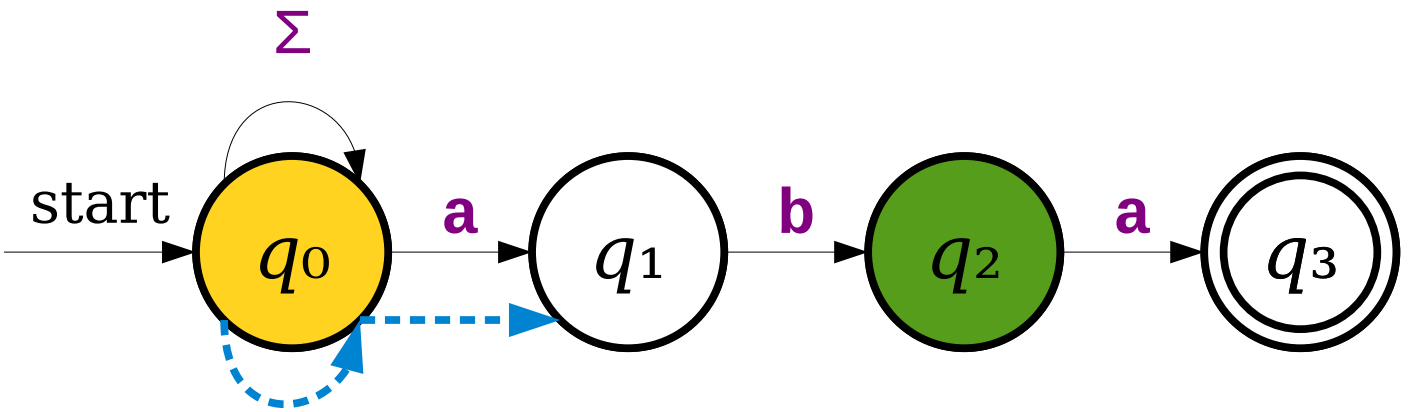

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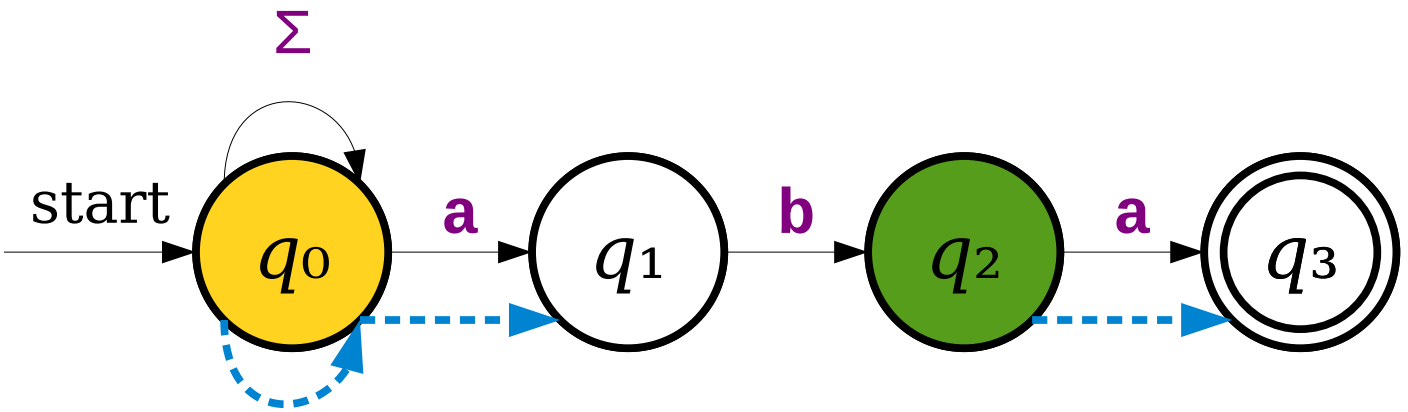

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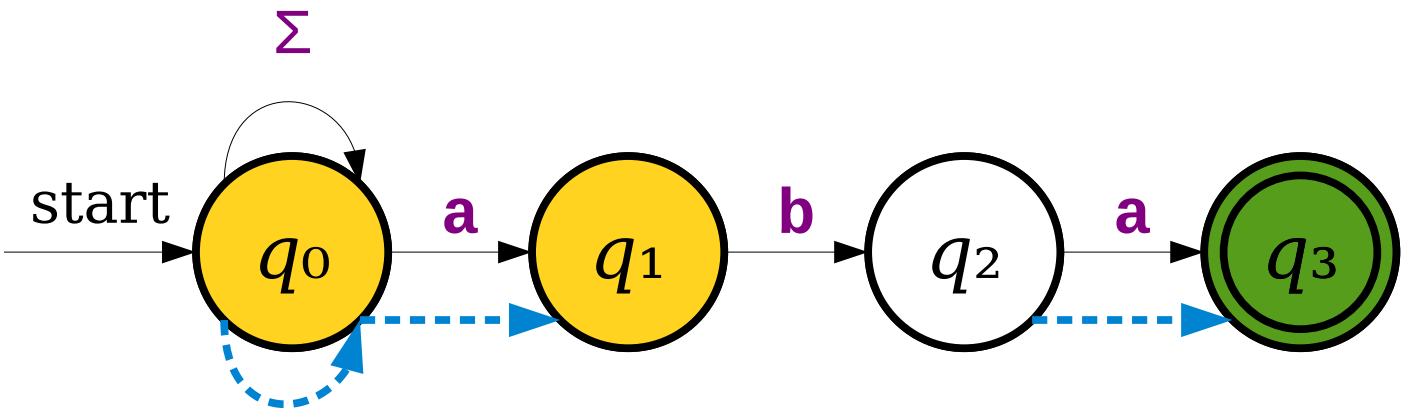

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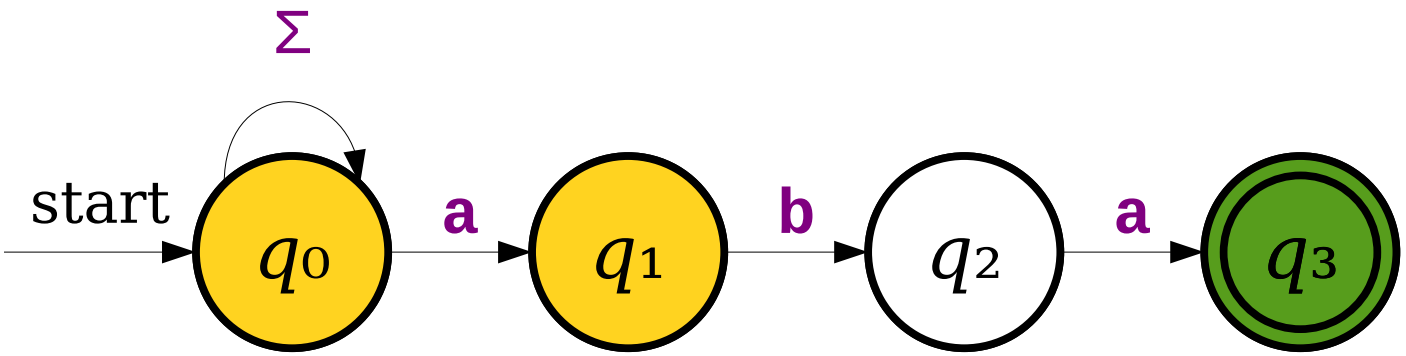

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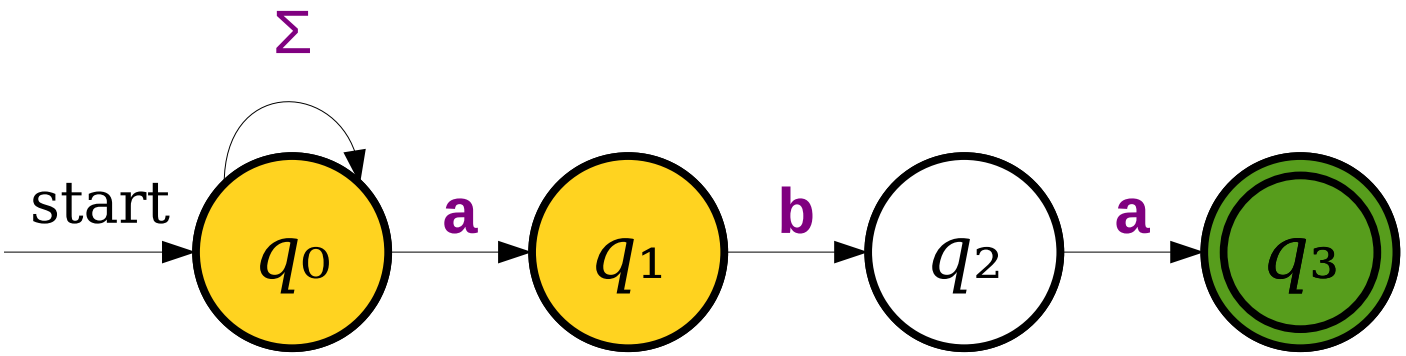

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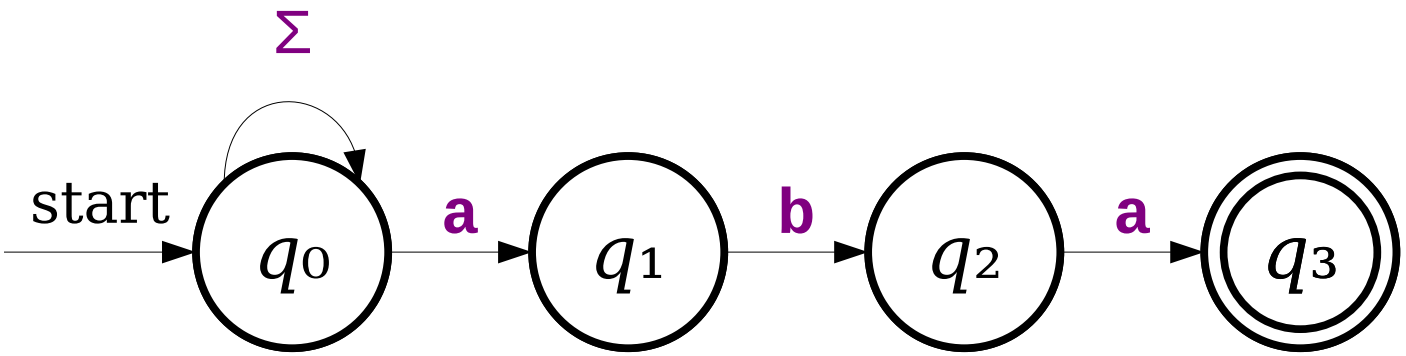

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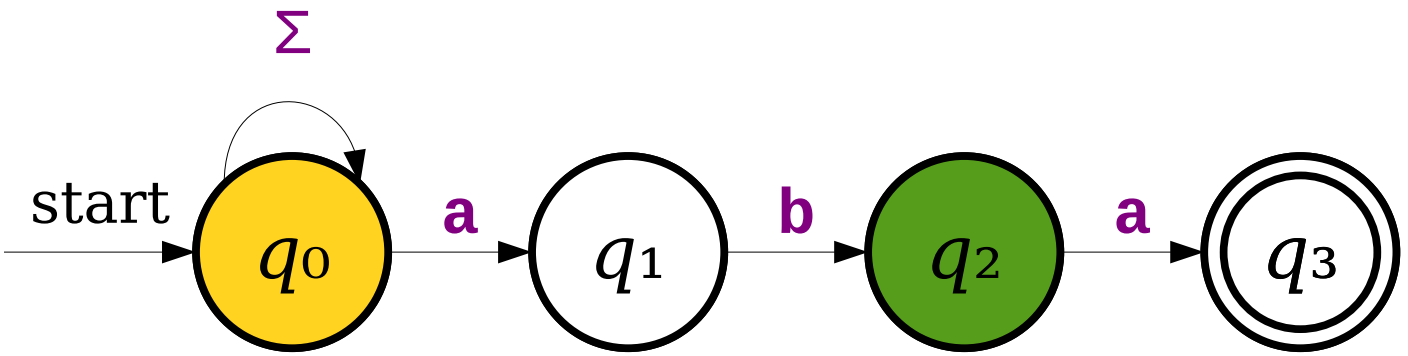

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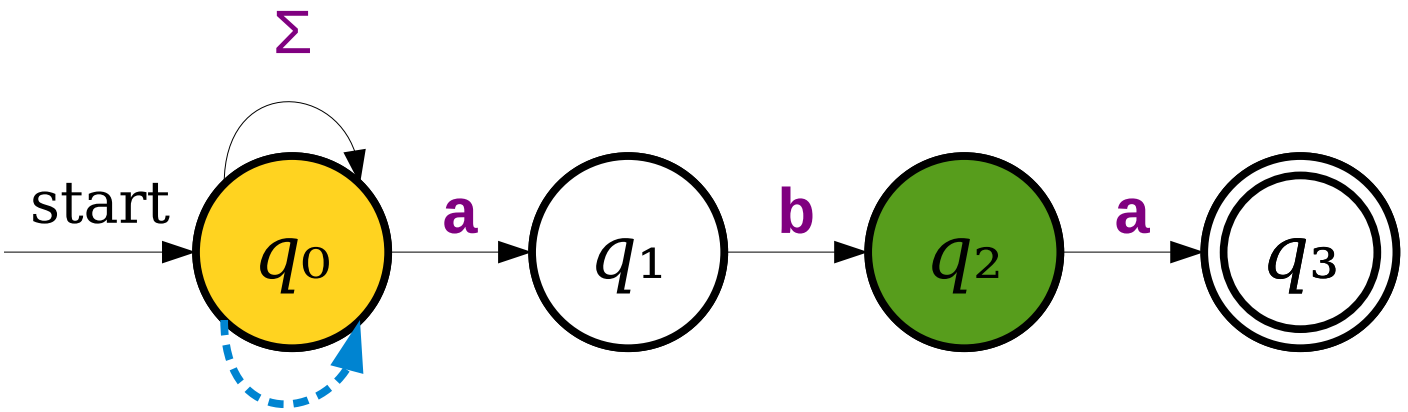

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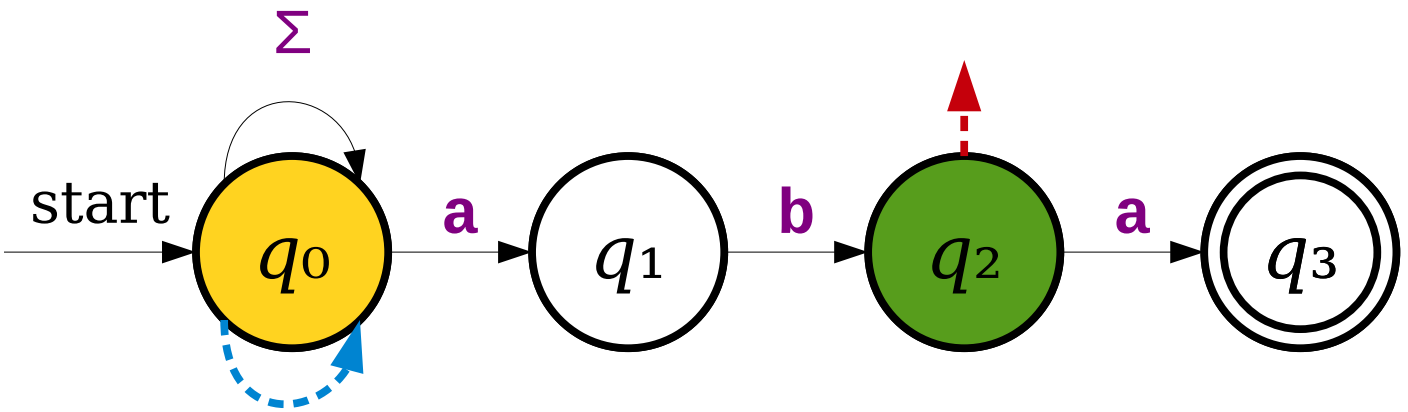

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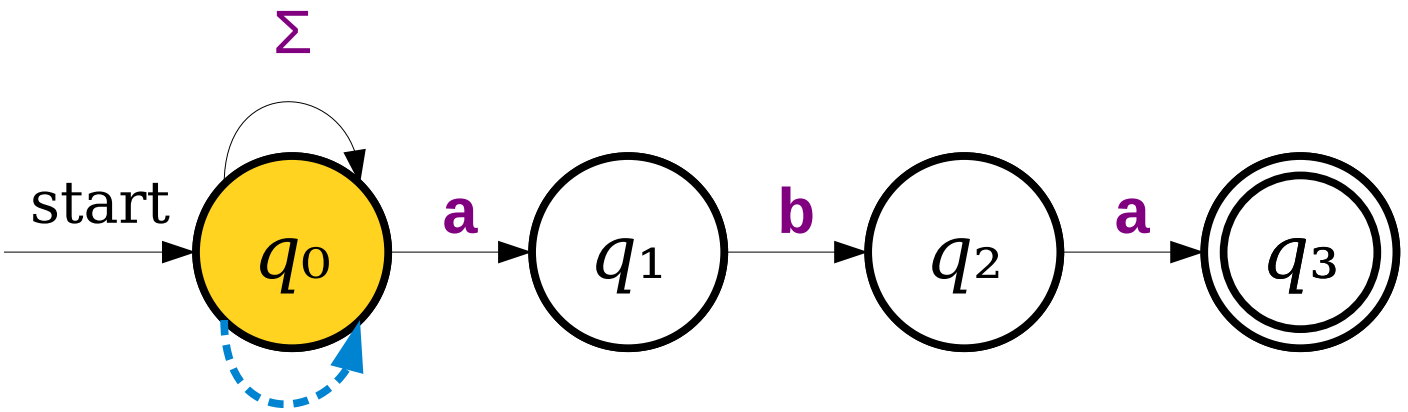

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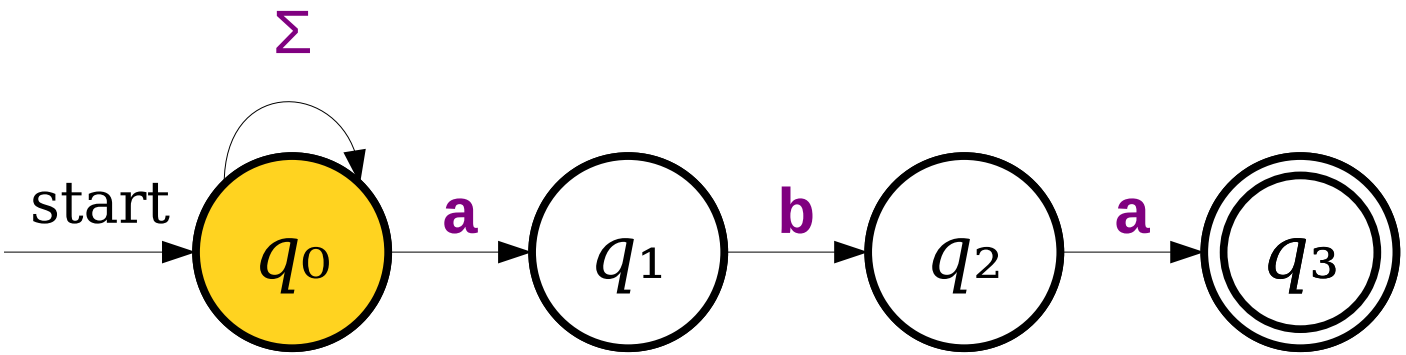

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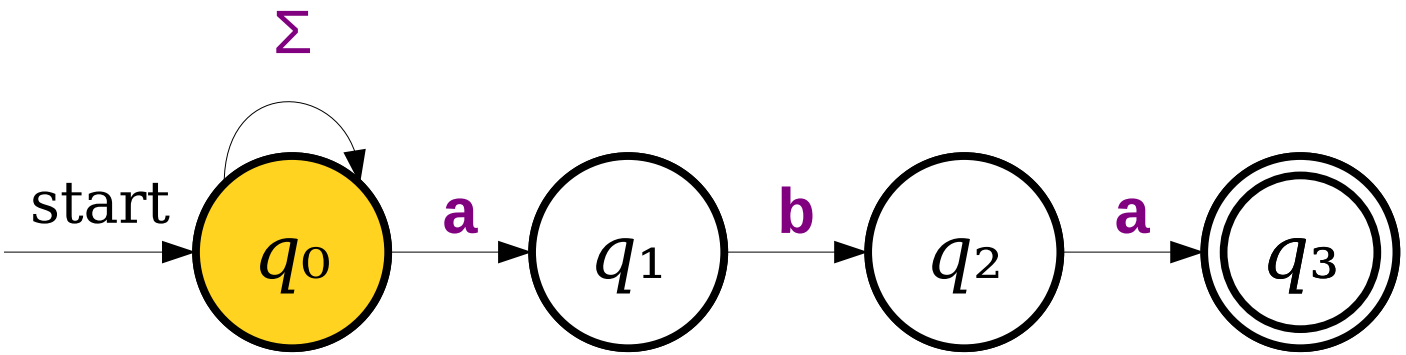

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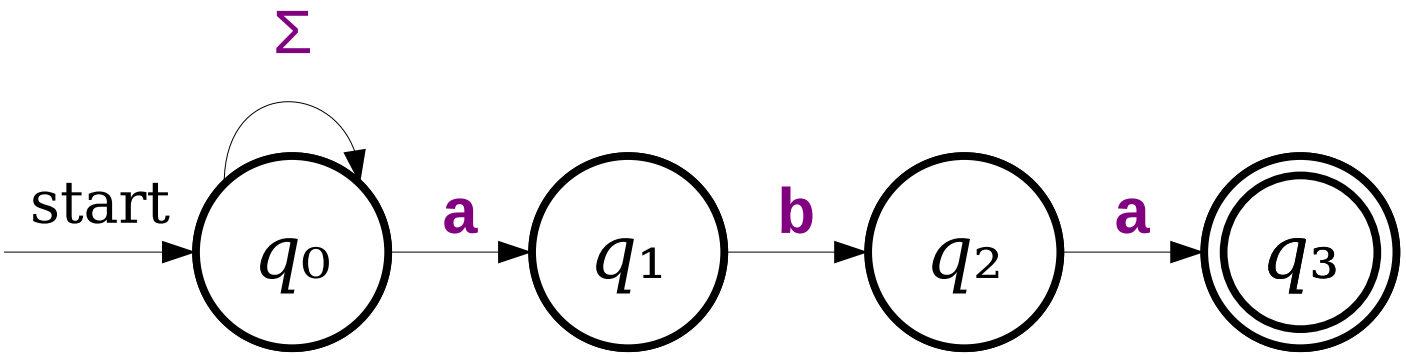

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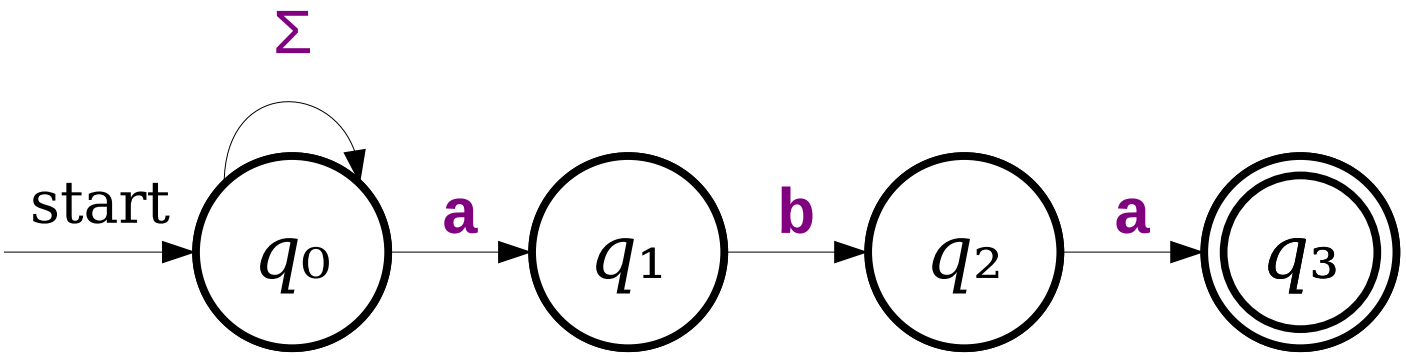

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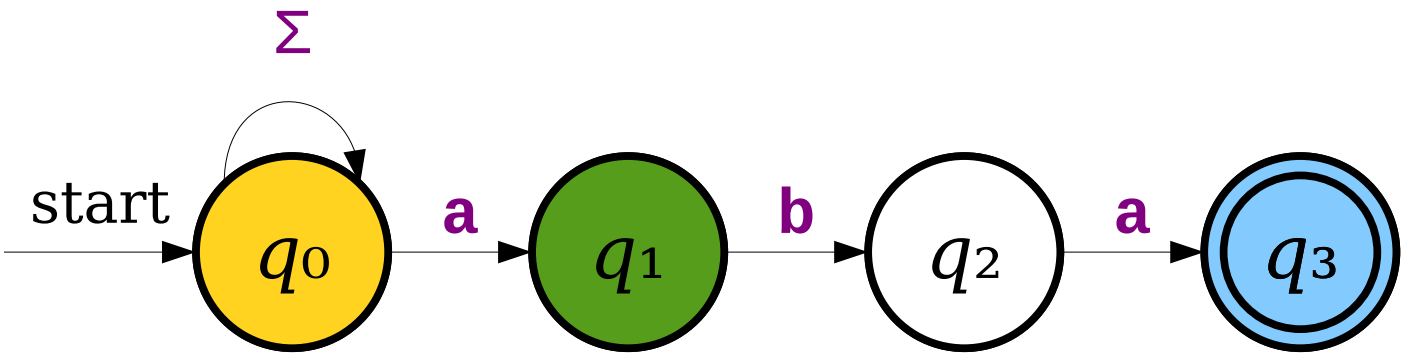

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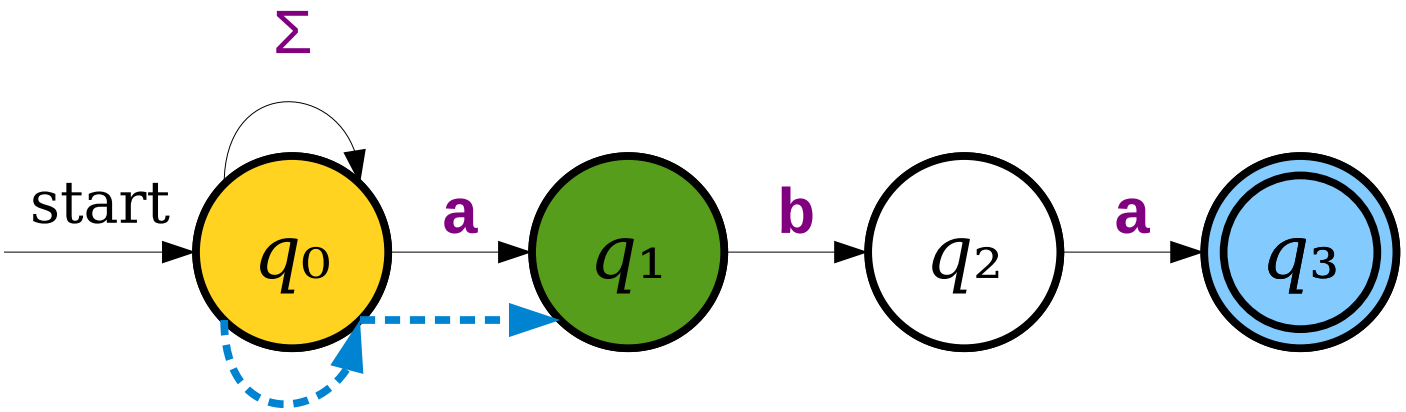

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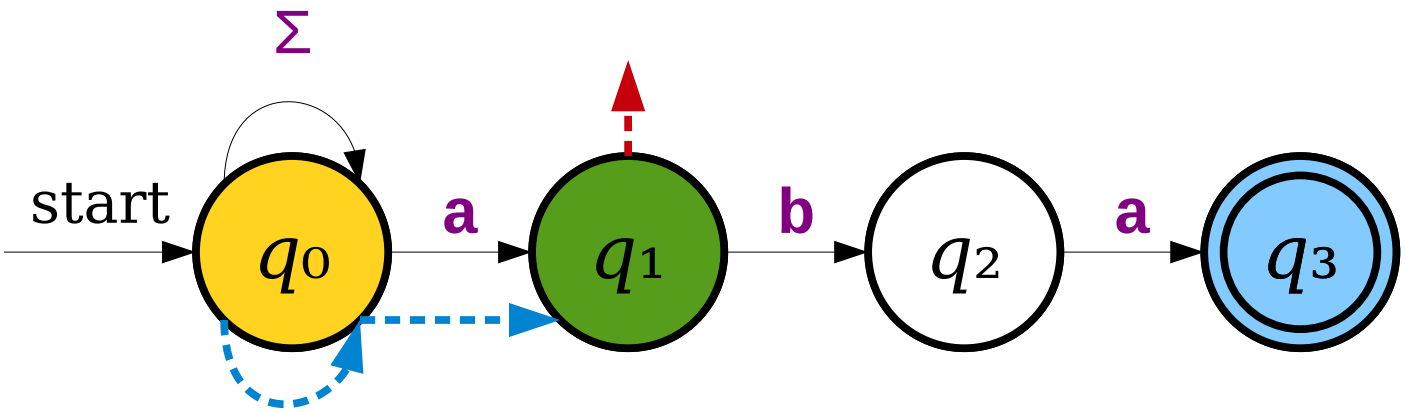

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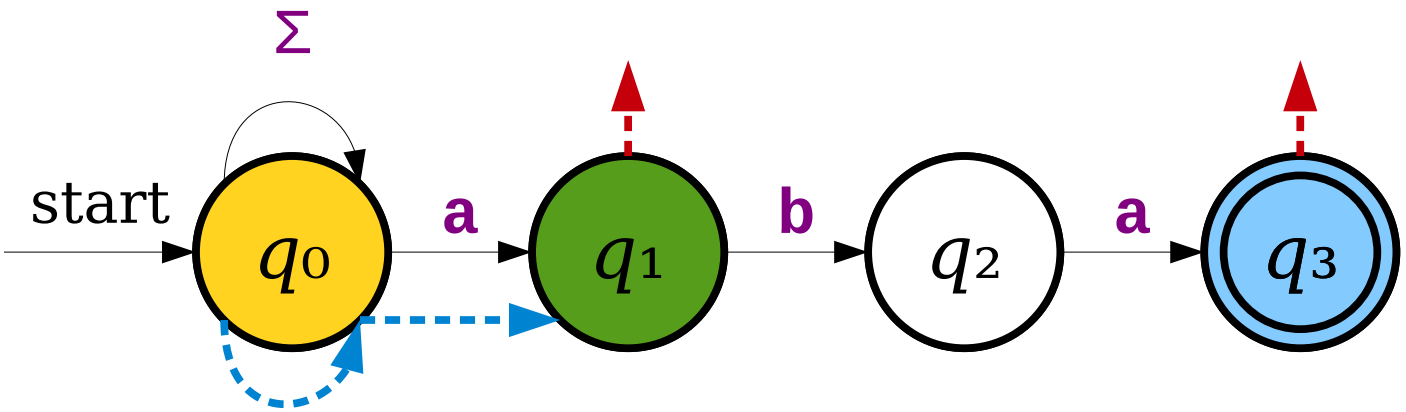

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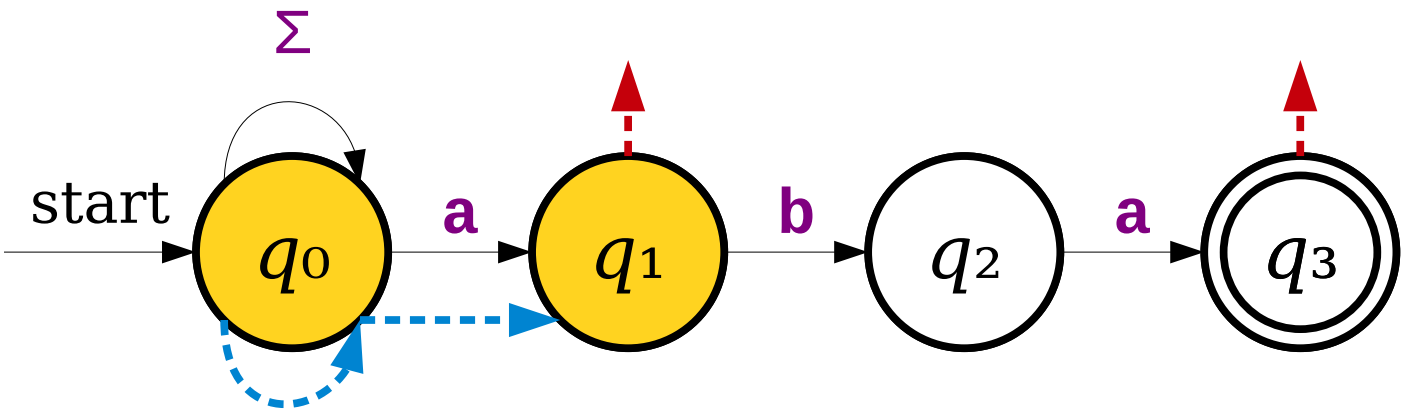

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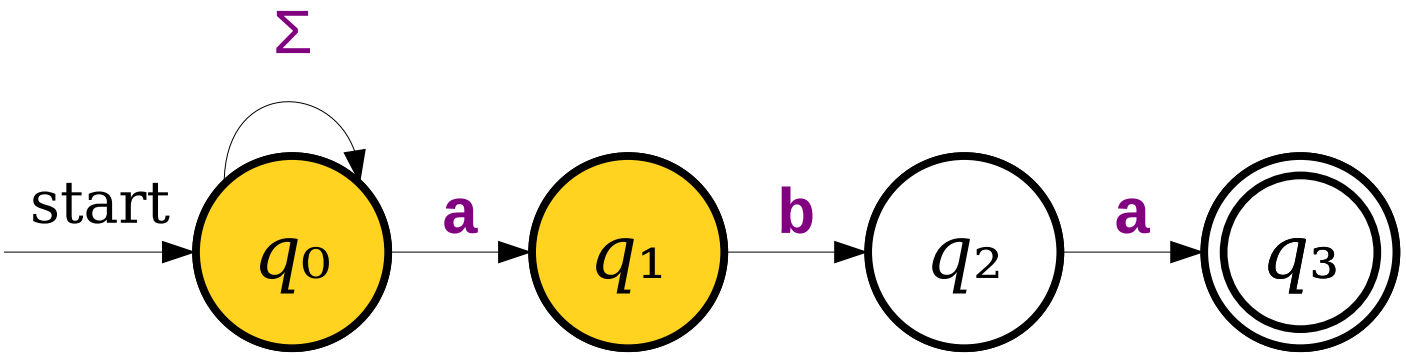

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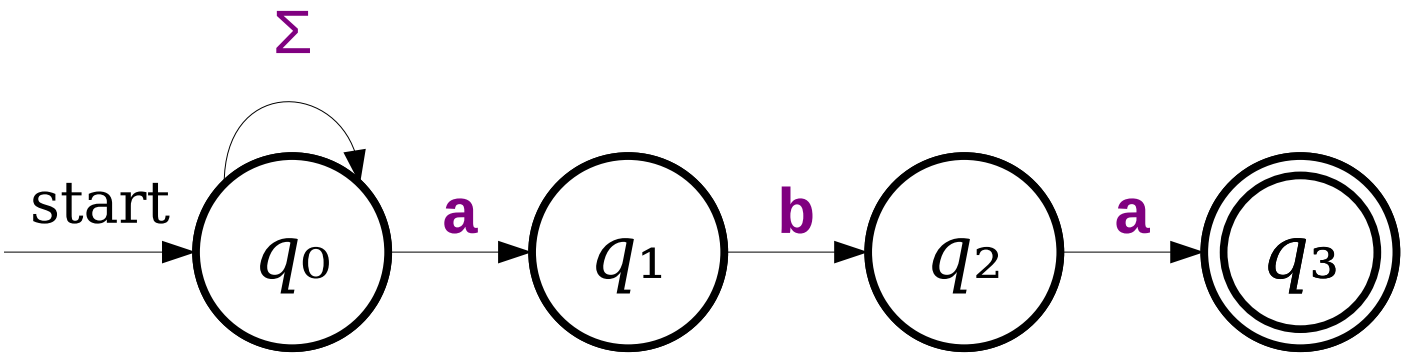

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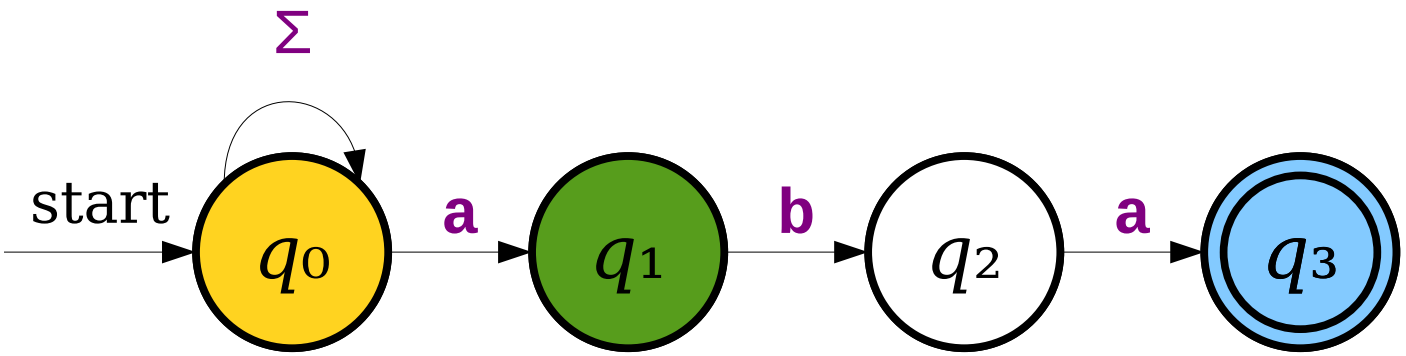

	a	b
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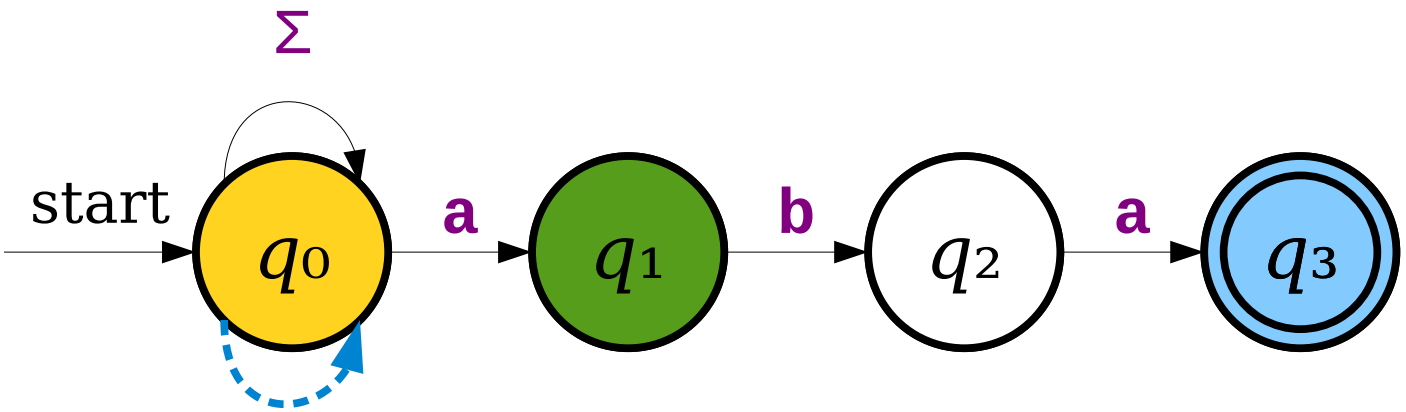

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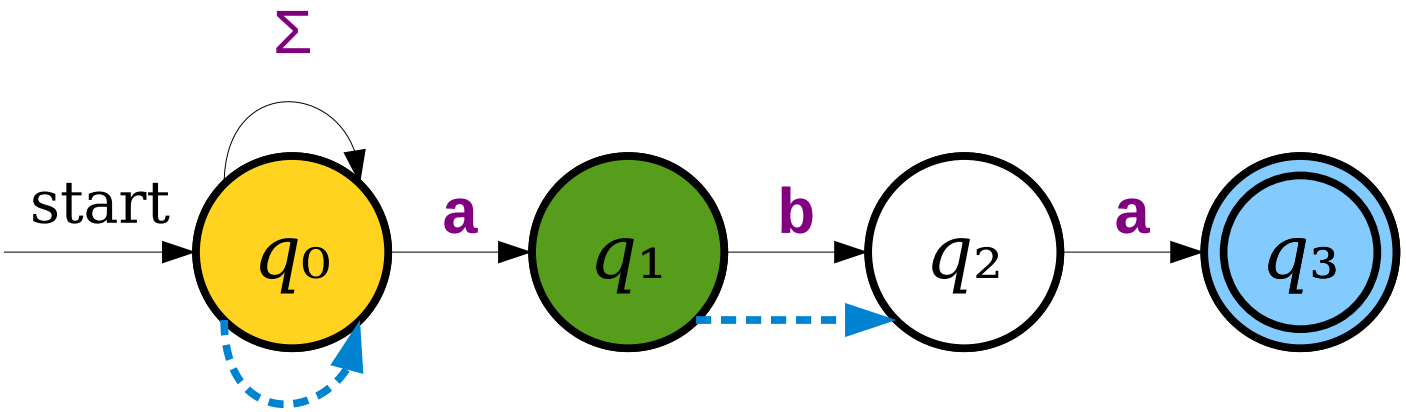

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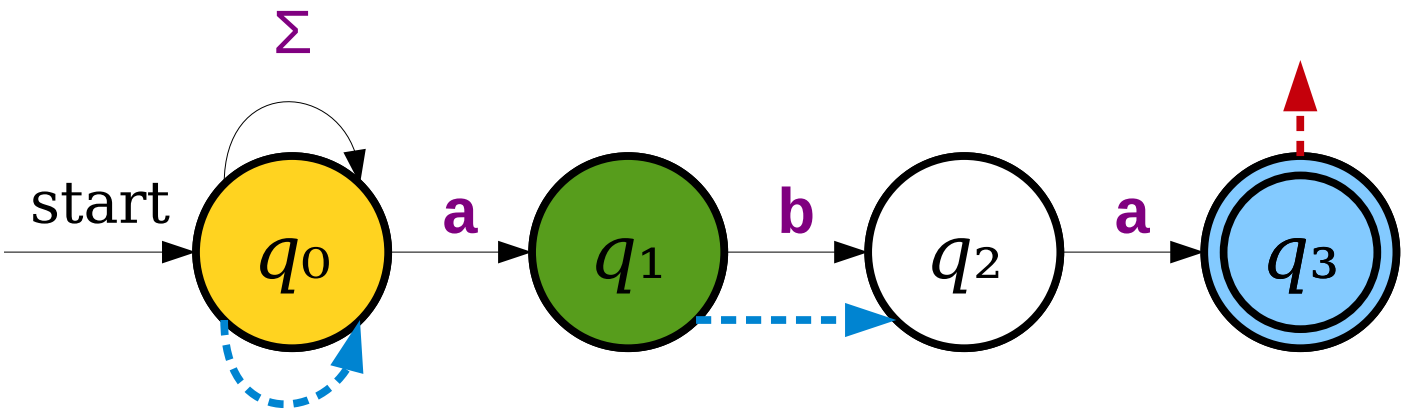

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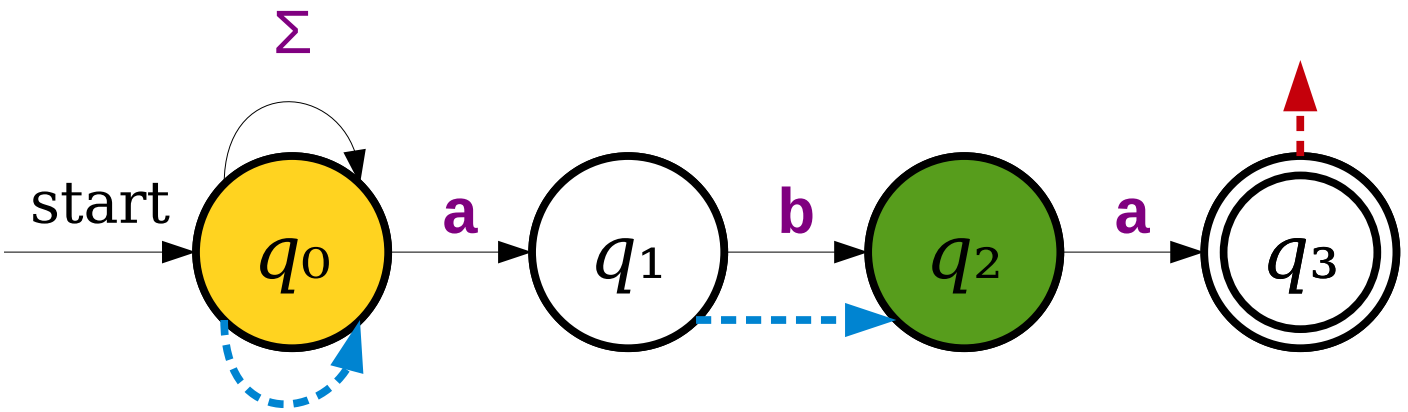

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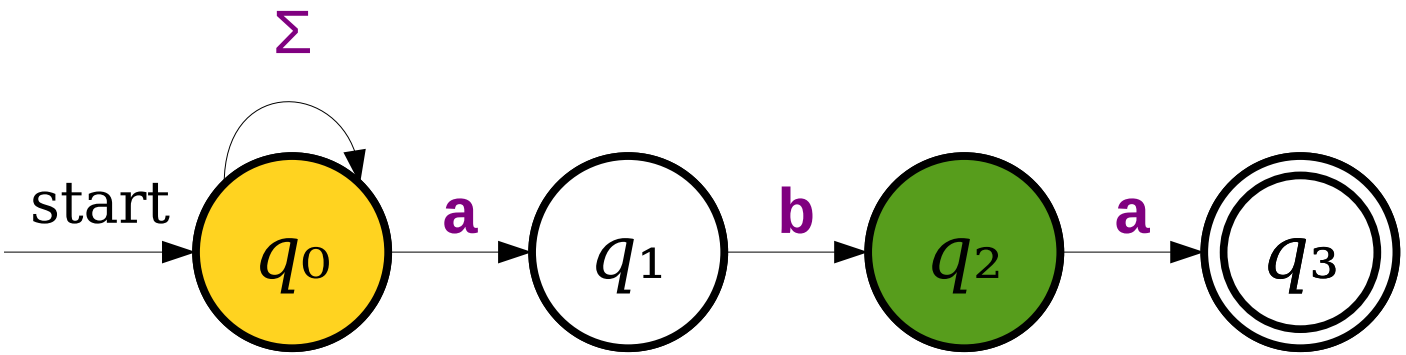

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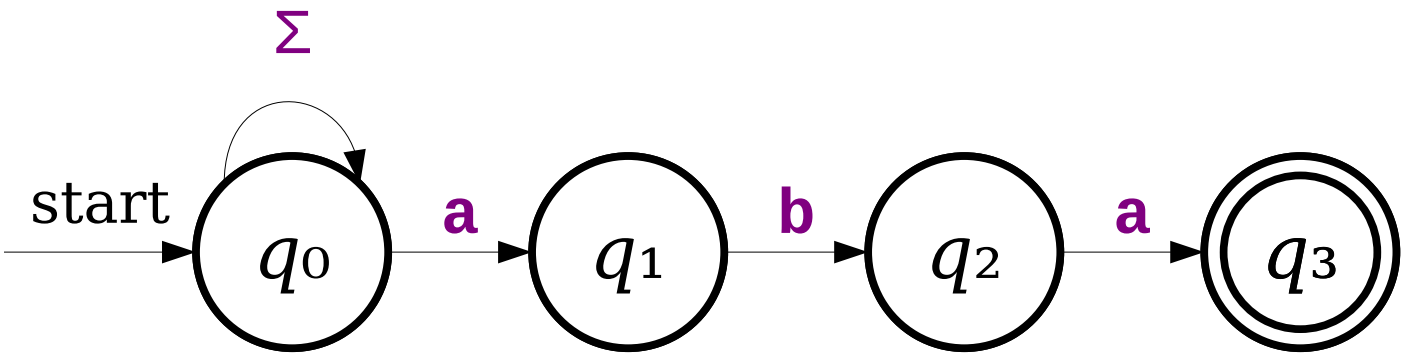

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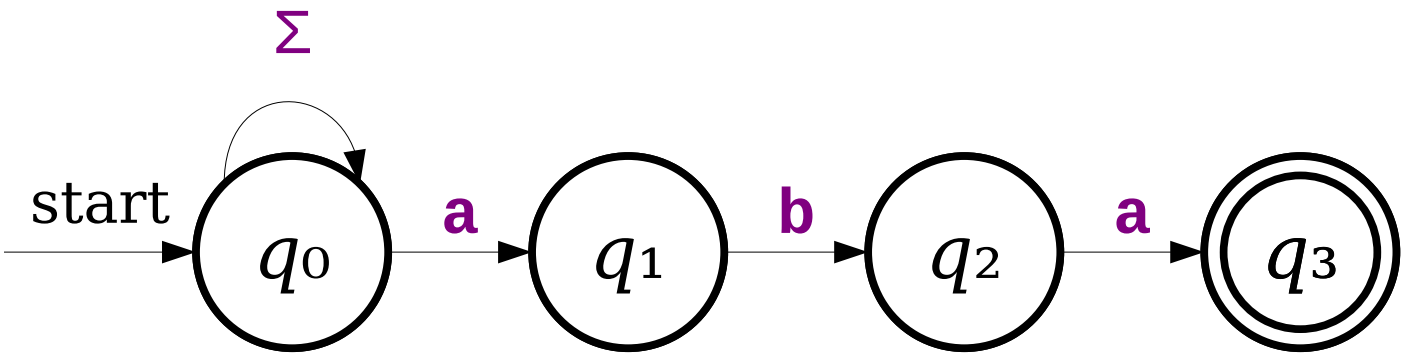

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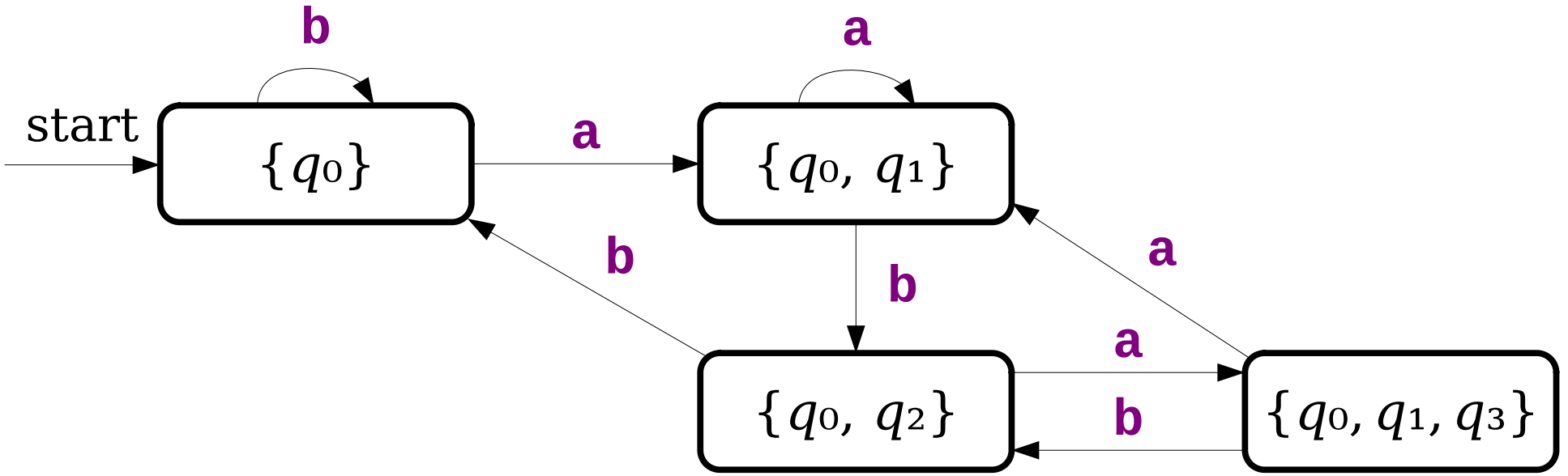

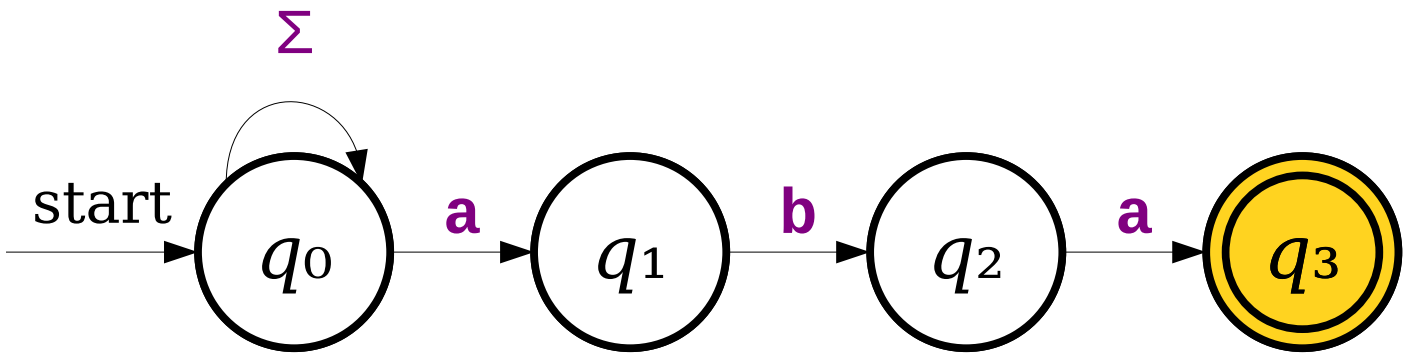

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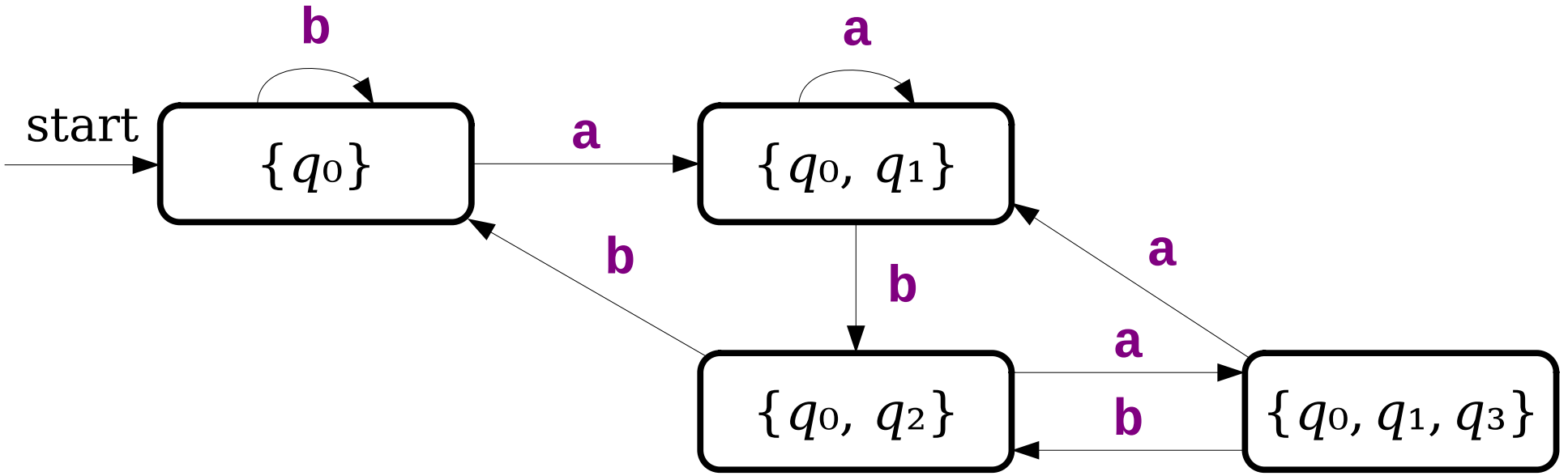

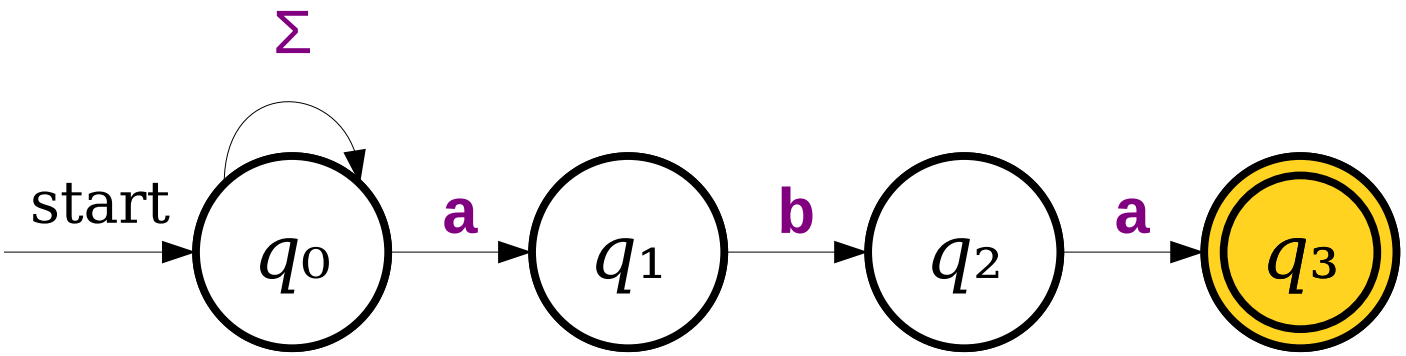

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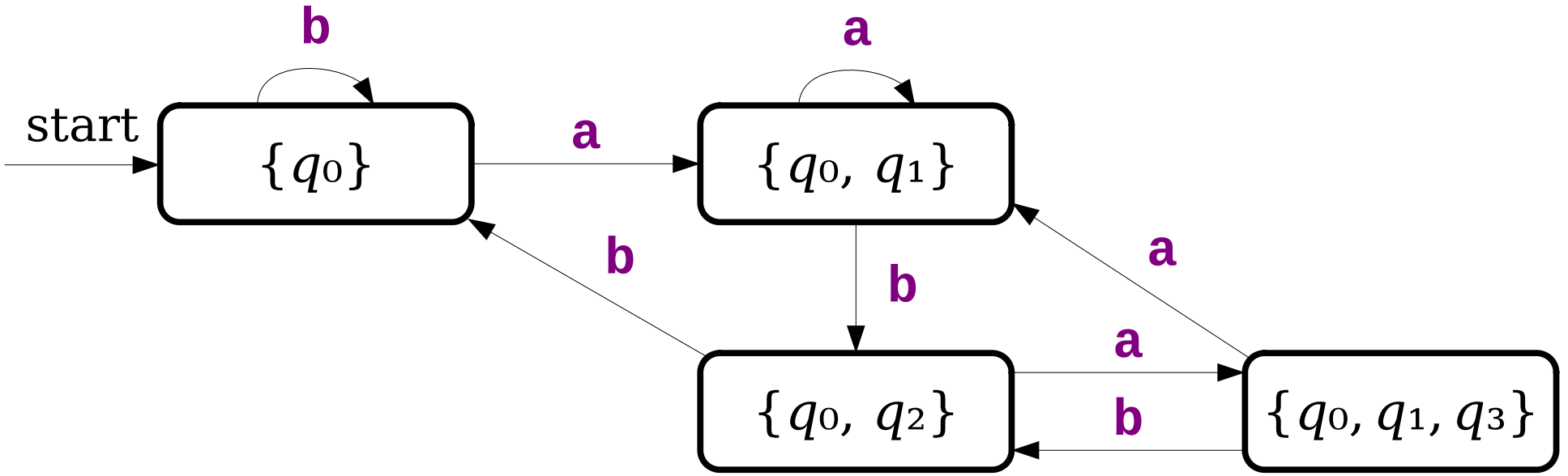

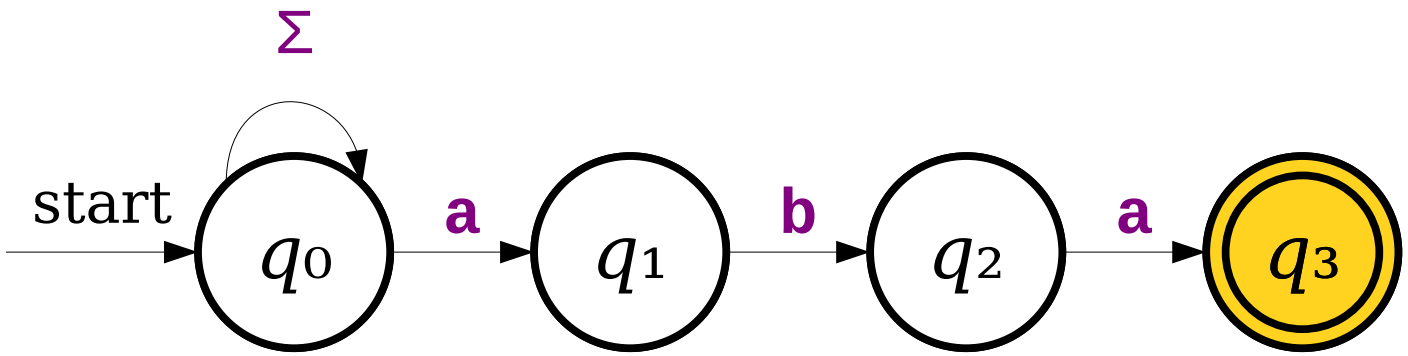

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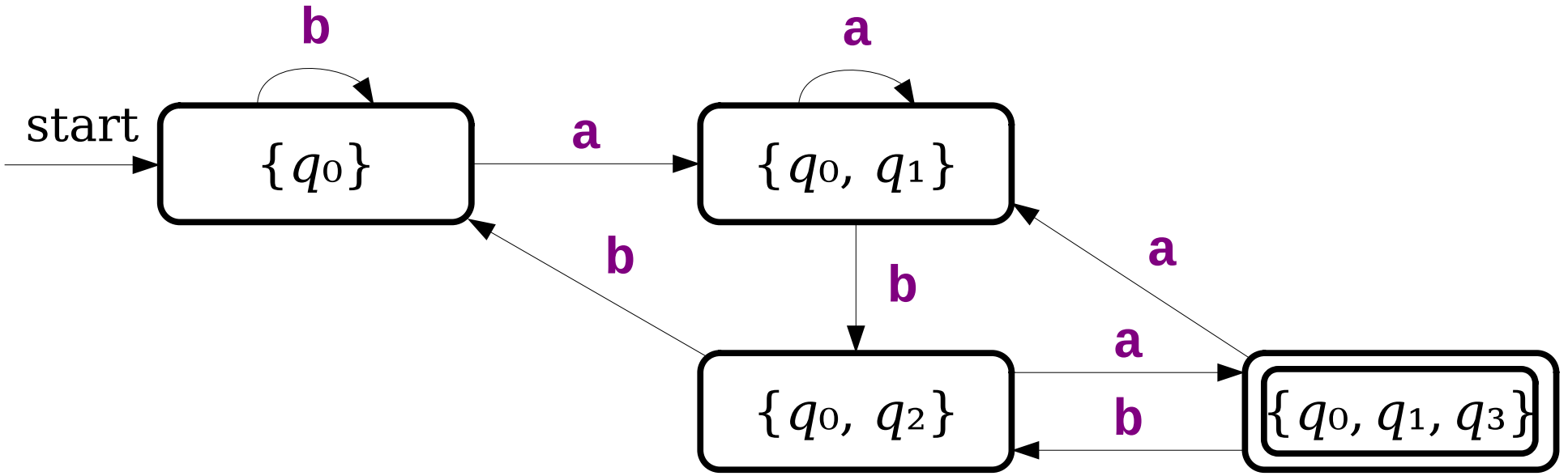

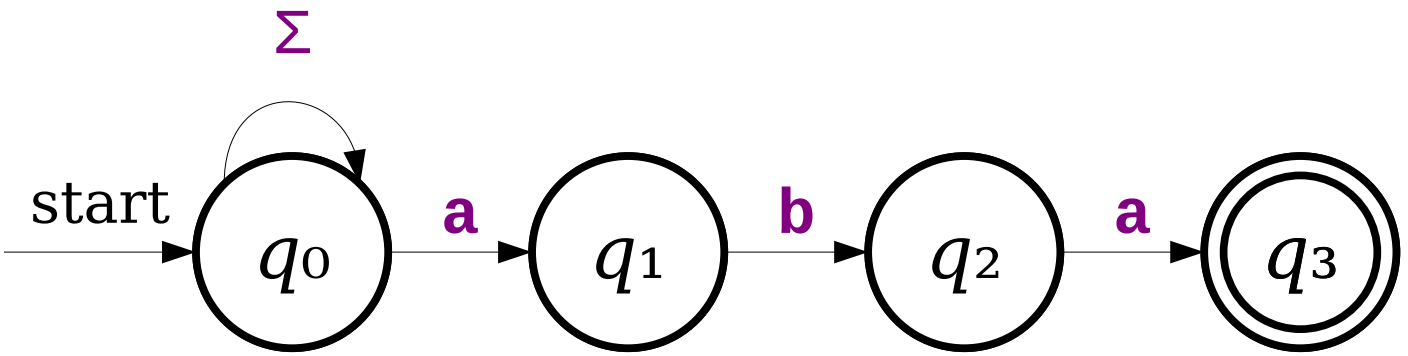

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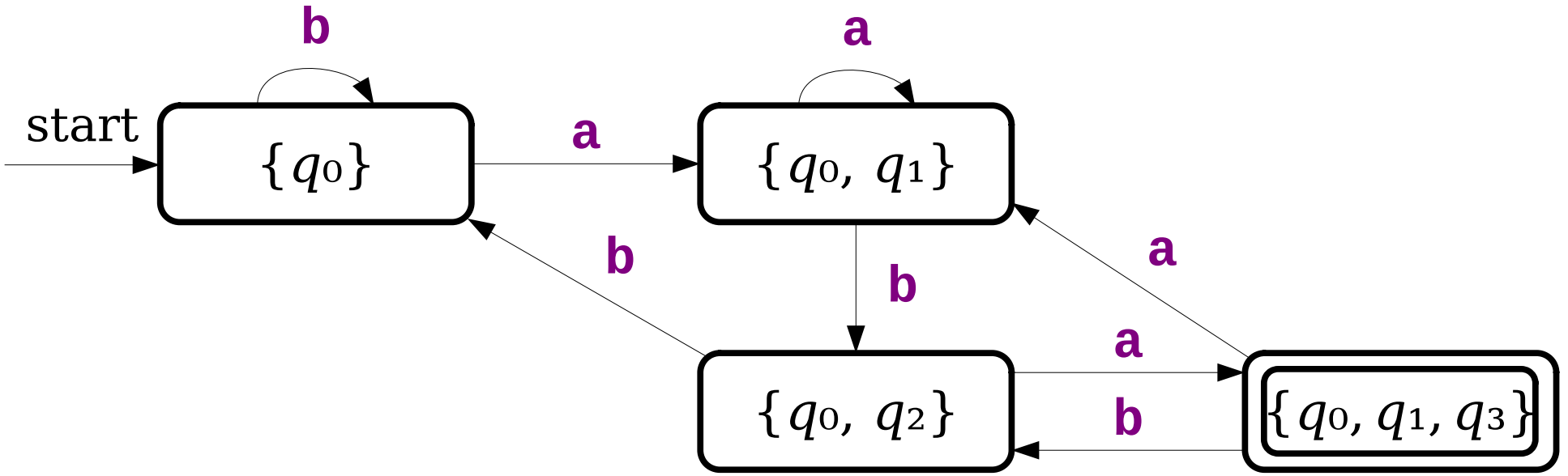

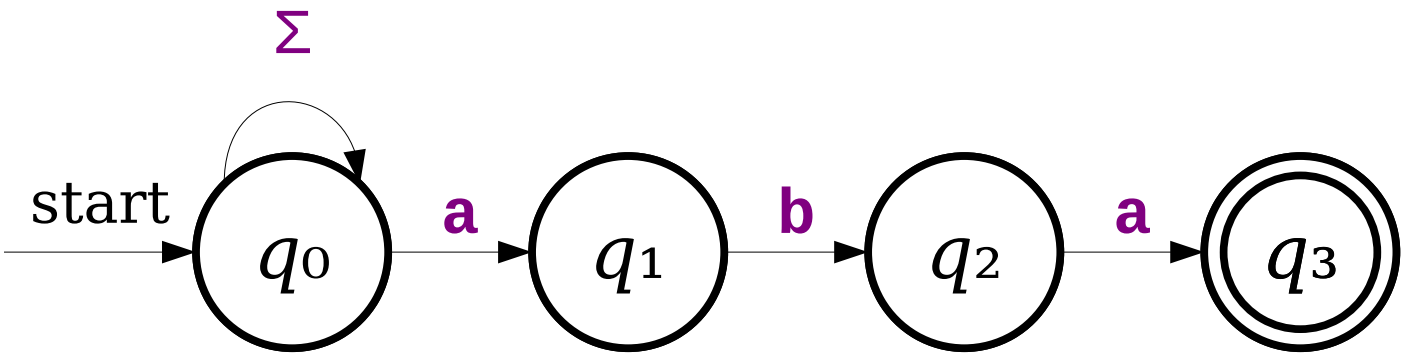

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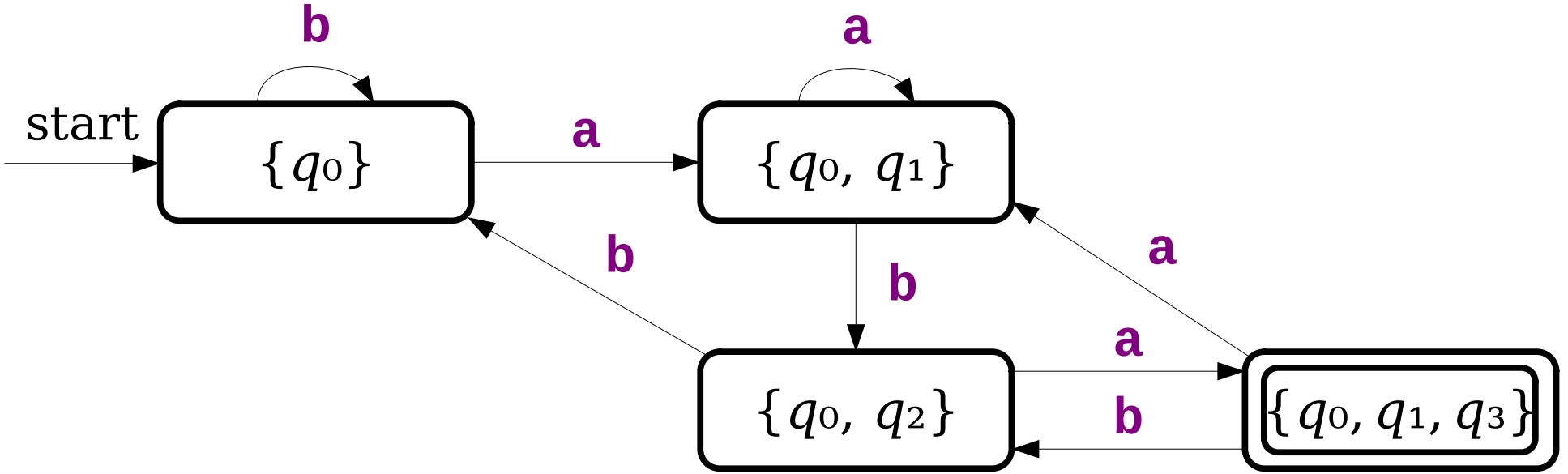

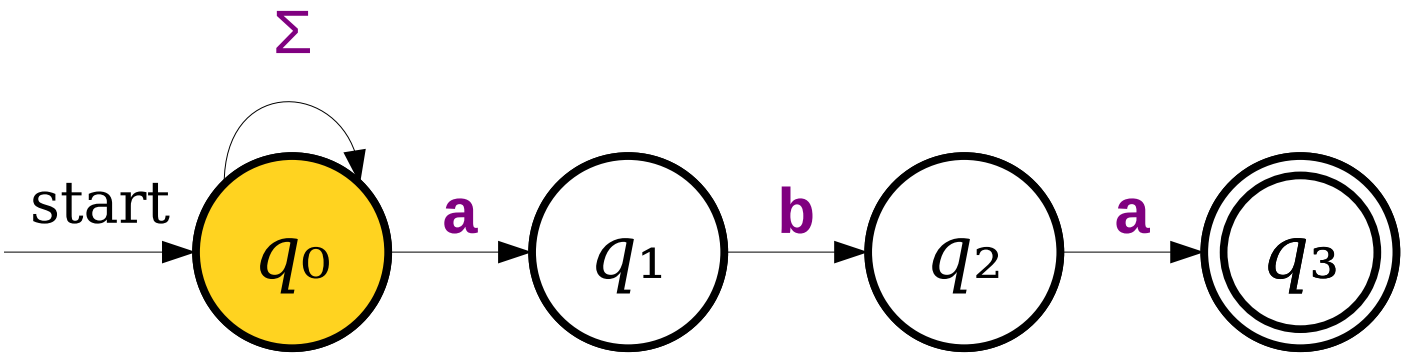

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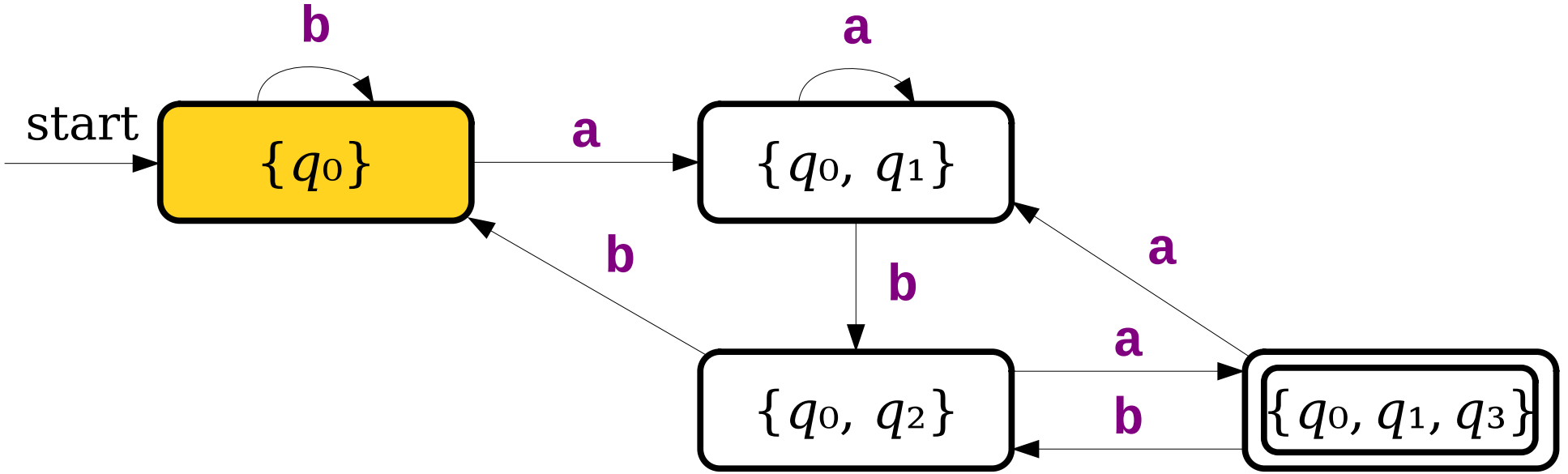

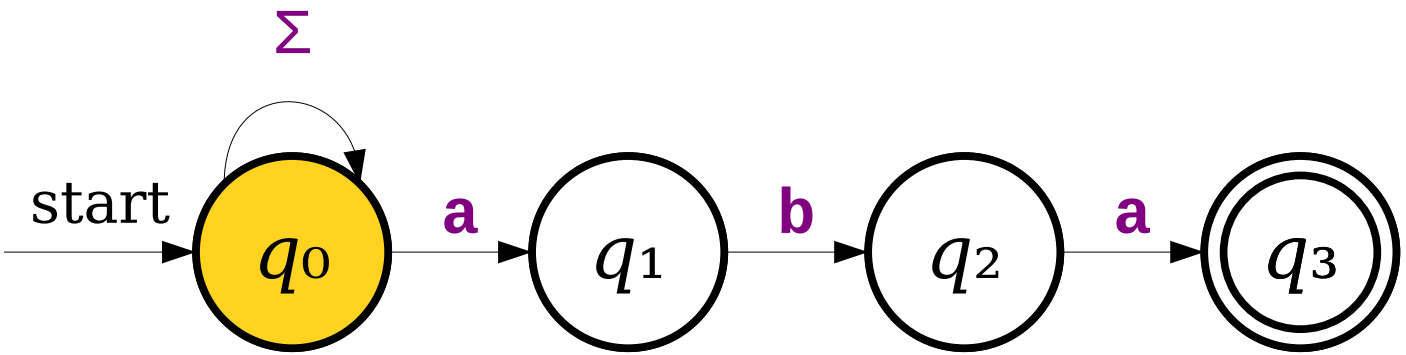
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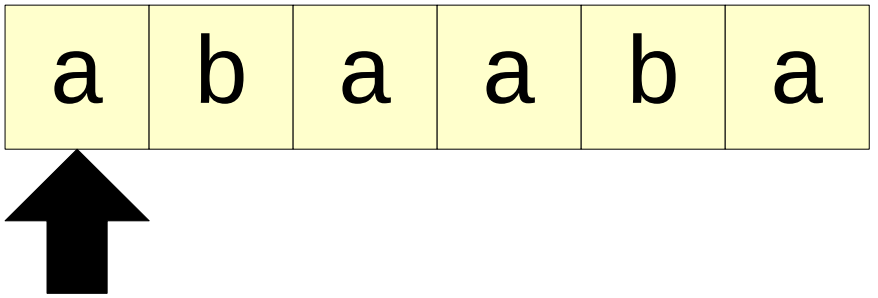
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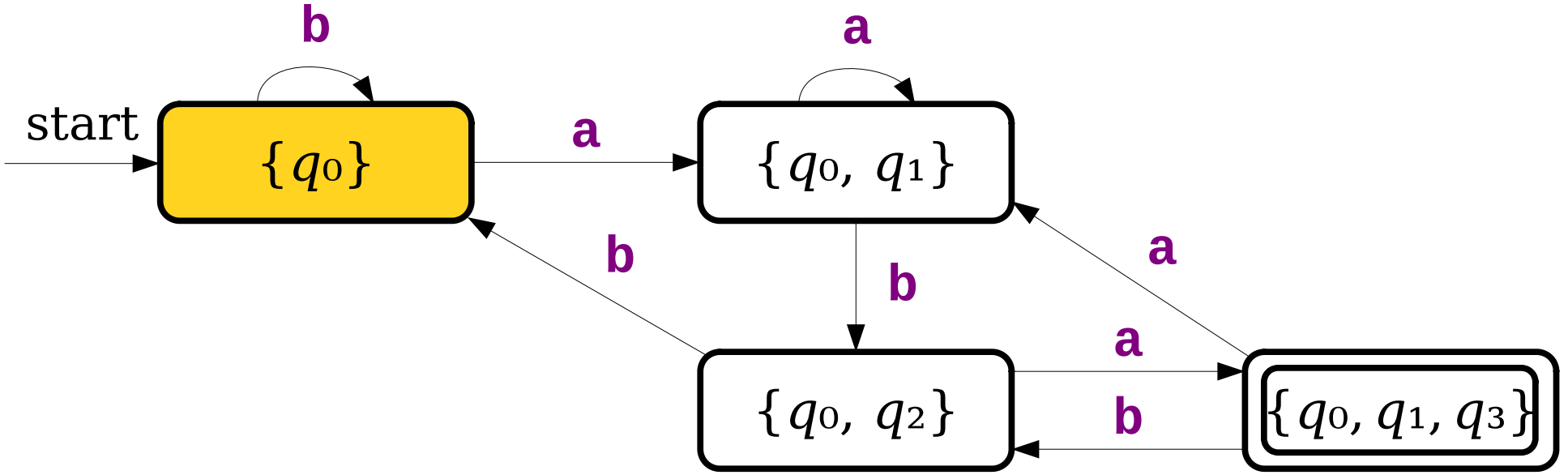
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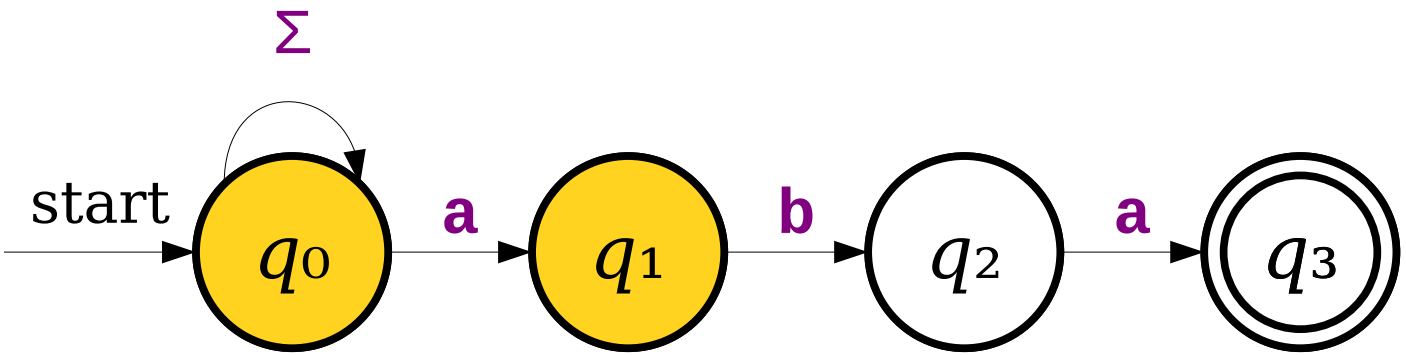
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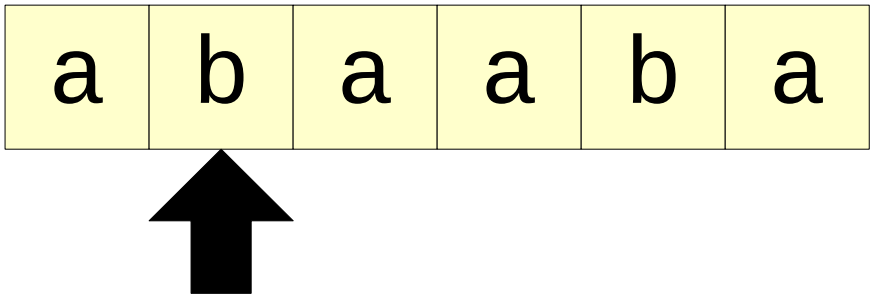
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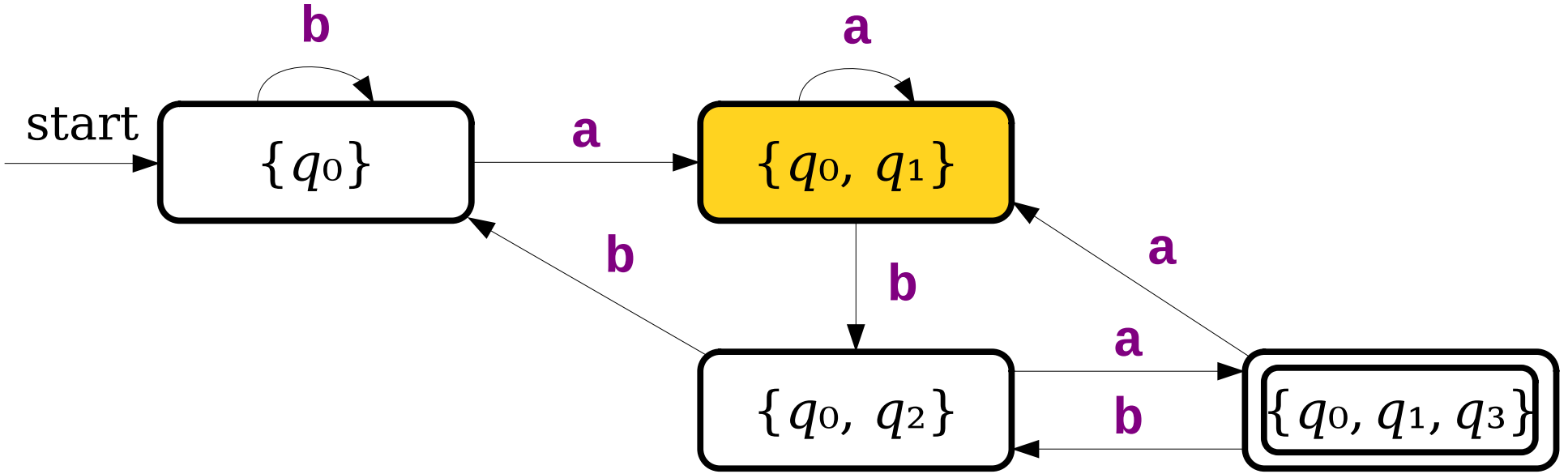
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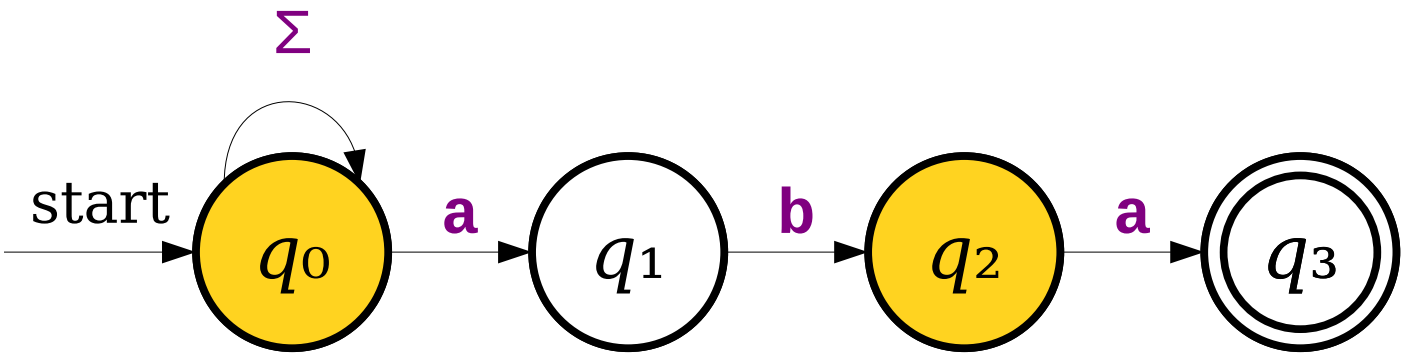
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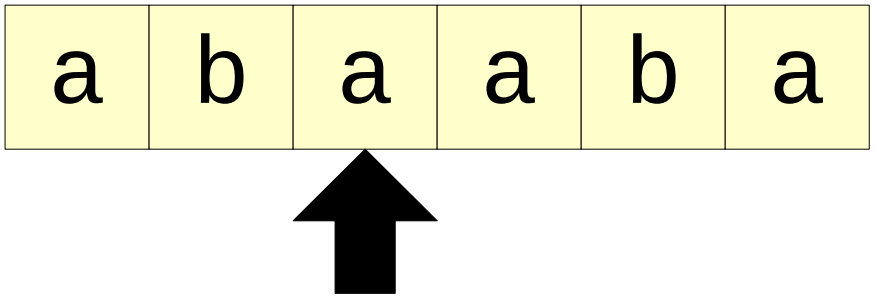
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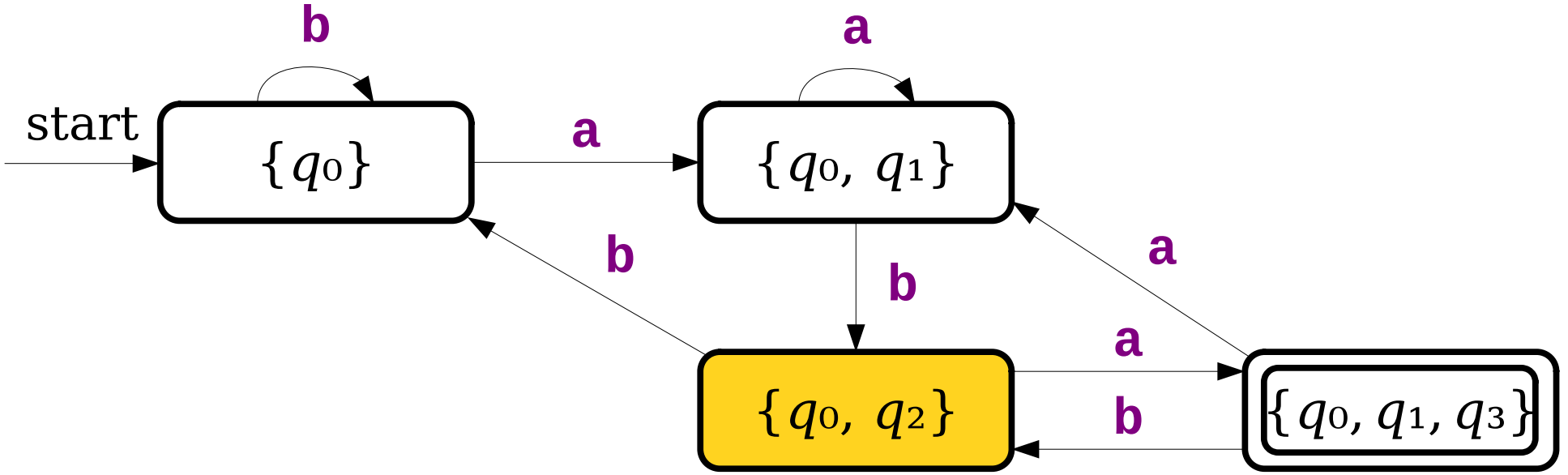
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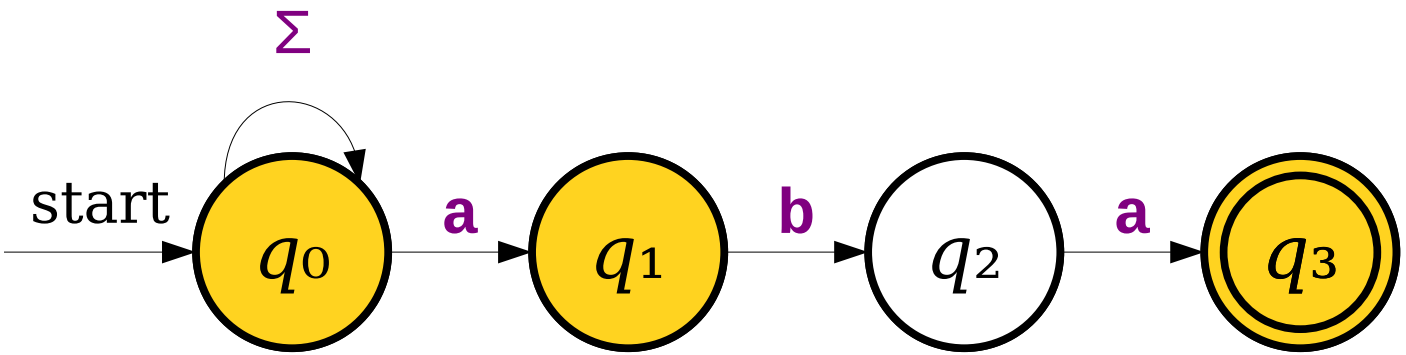
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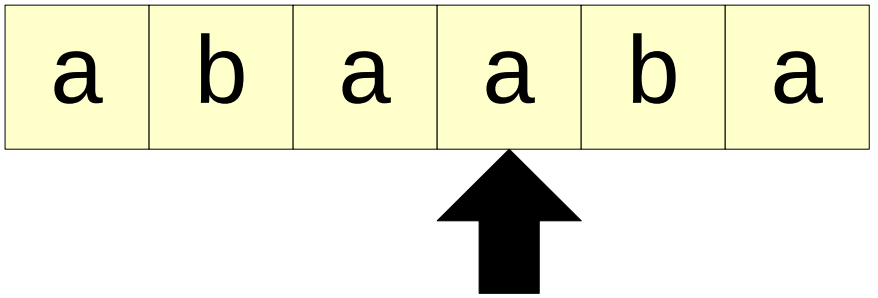
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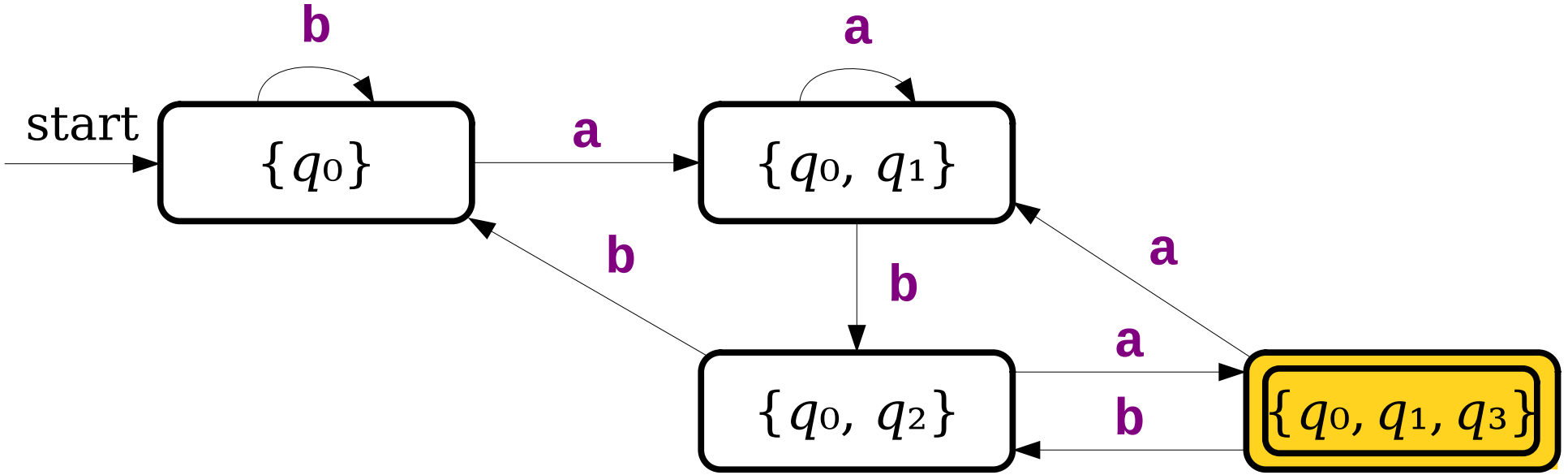
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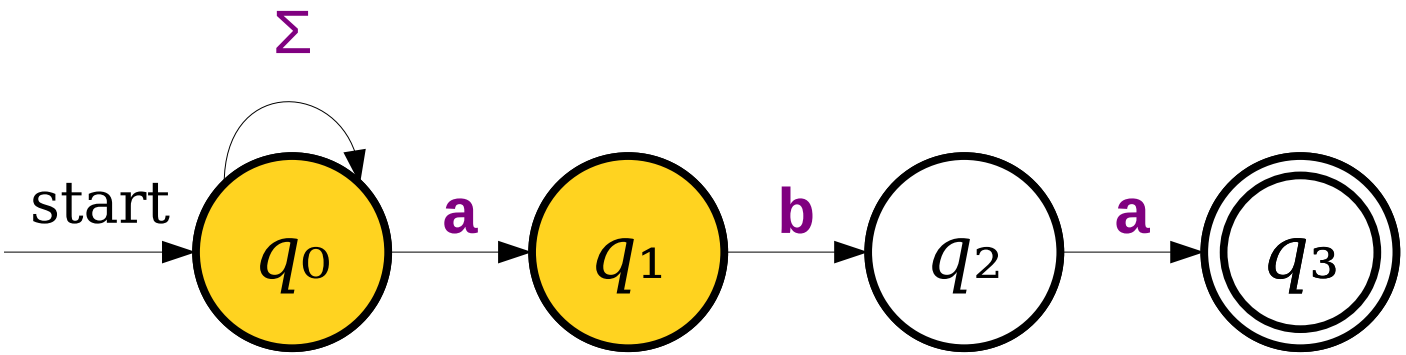
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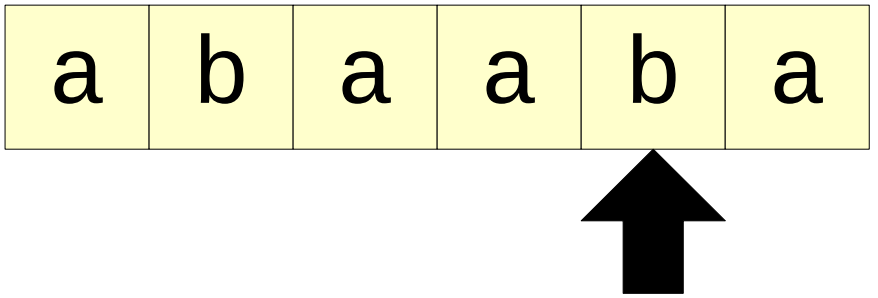
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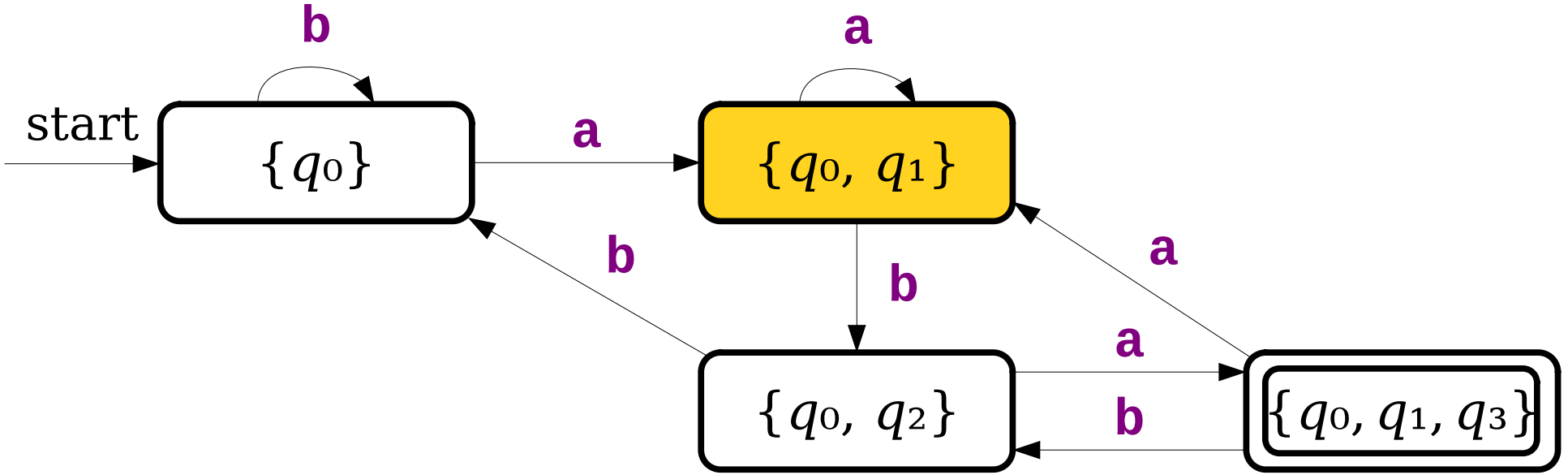
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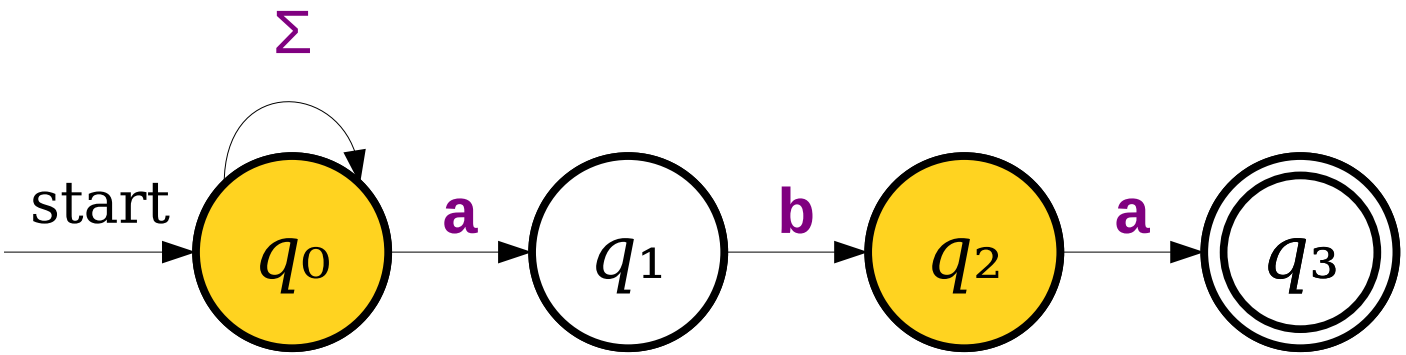
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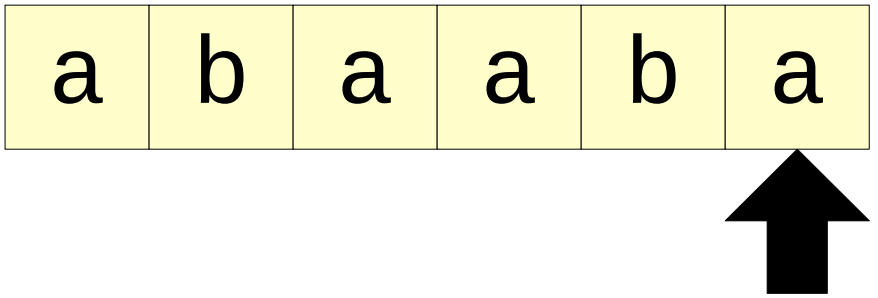
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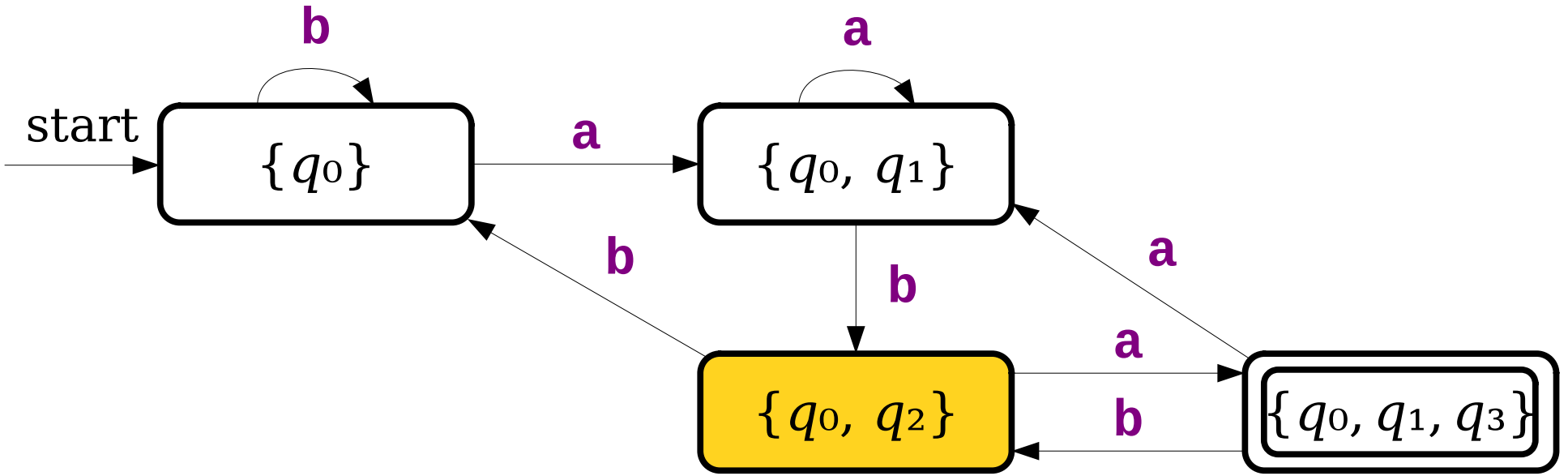
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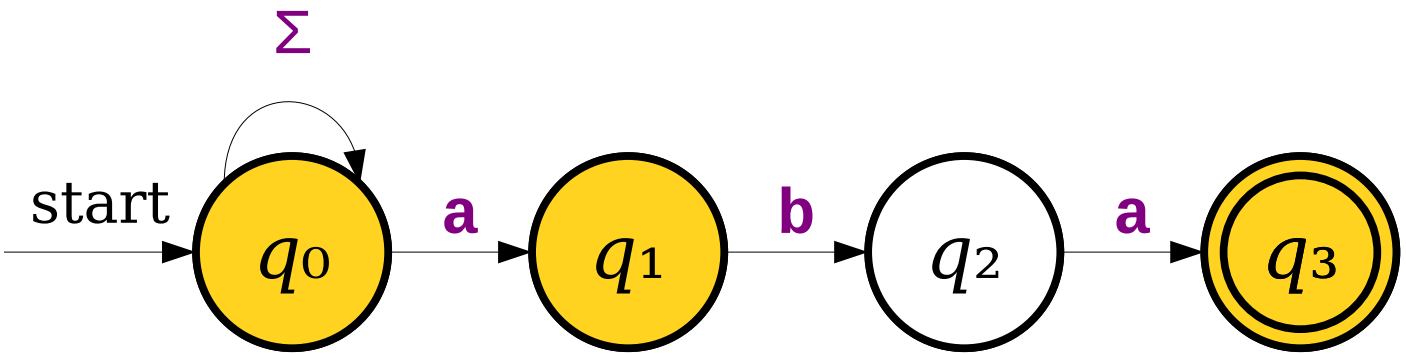

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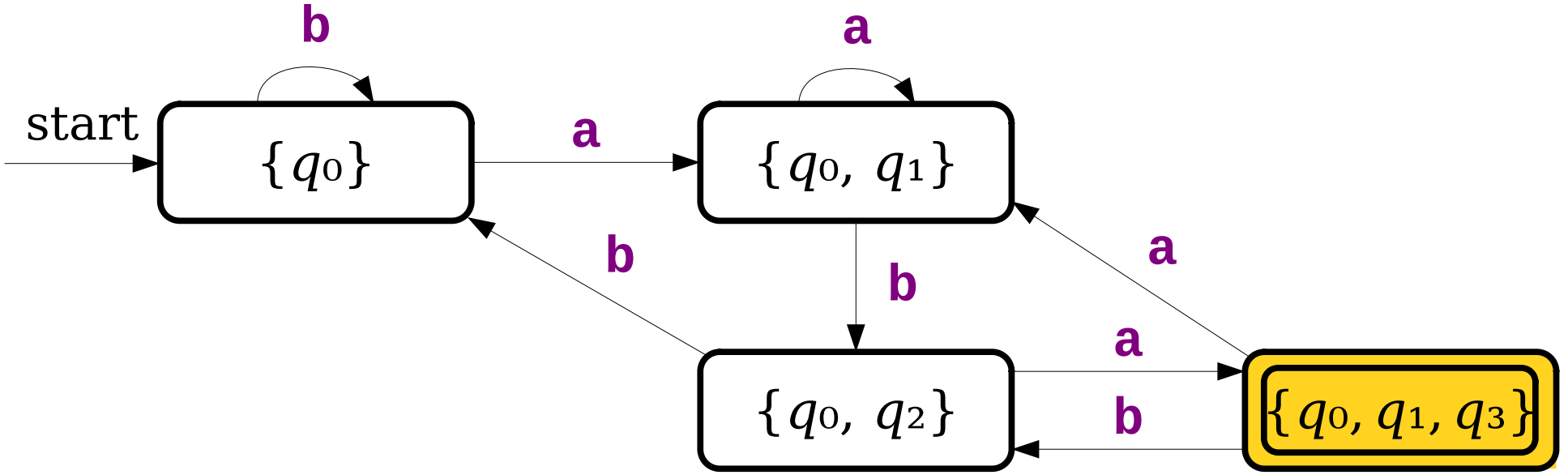

The Subset Construction

- This procedure for turning an NFA for a language L into a DFA for a language L is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size n, but no DFAs of size less than 2^n?

A language L is called a ***regular language*** if there exists a DFA D such that $\mathscr{L}(D) = L$.

An Important Result

Theorem: A language L is regular if and only if there is some NFA N such that $\mathscr{L}(N) = L$.

Proof Sketch: Pick a language L. First, assume L is regular. That means there's a DFA D where $\mathscr{L}(D) = L$. Every DFA is "basically" an NFA, so there's an NFA (D) whose language is L.

Next, assume there's an NFA N such that $\mathscr{L}(N) = L$. Using the subset construction, we can build a DFA D where $\mathscr{L}(N) = \mathscr{L}(D)$. Then we have that $\mathscr{L}(D) = L$, so L is regular. ■-*ish*

Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

Properties of Regular Languages

The Complement of a Language

- Given a language $L \subseteq \Sigma^*$, the ***complement*** of that language (denoted $\overline{L}$) is the language of all strings in Σ^* that aren't in L.
- Formally:

$$\overline{L} = \Sigma^* - L$$

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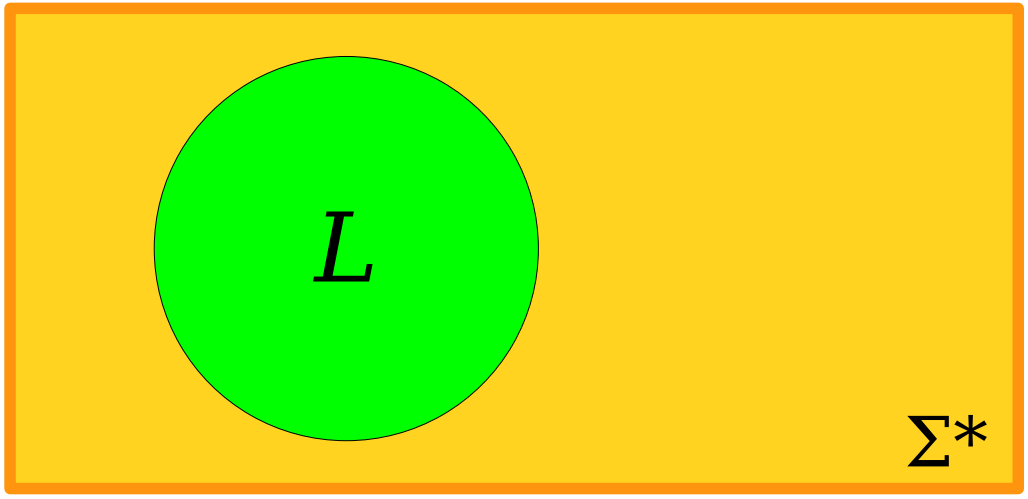

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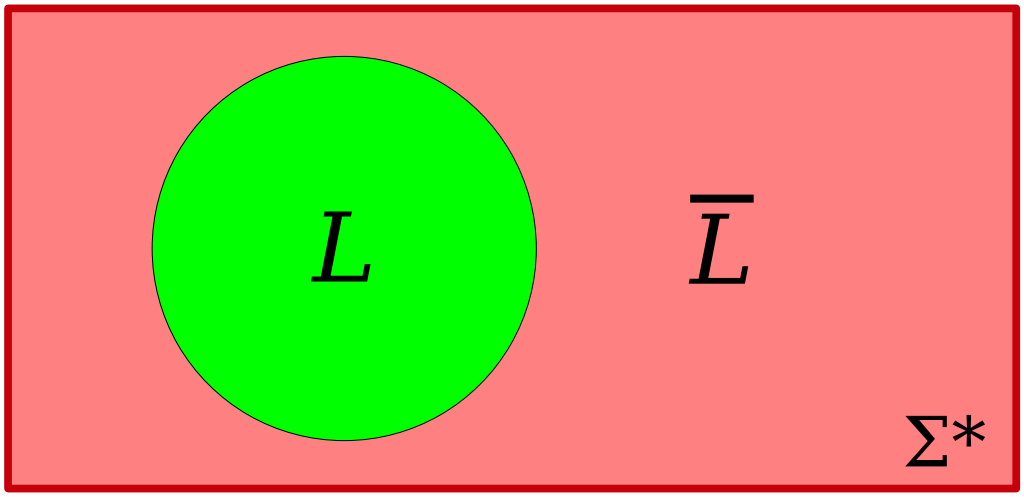

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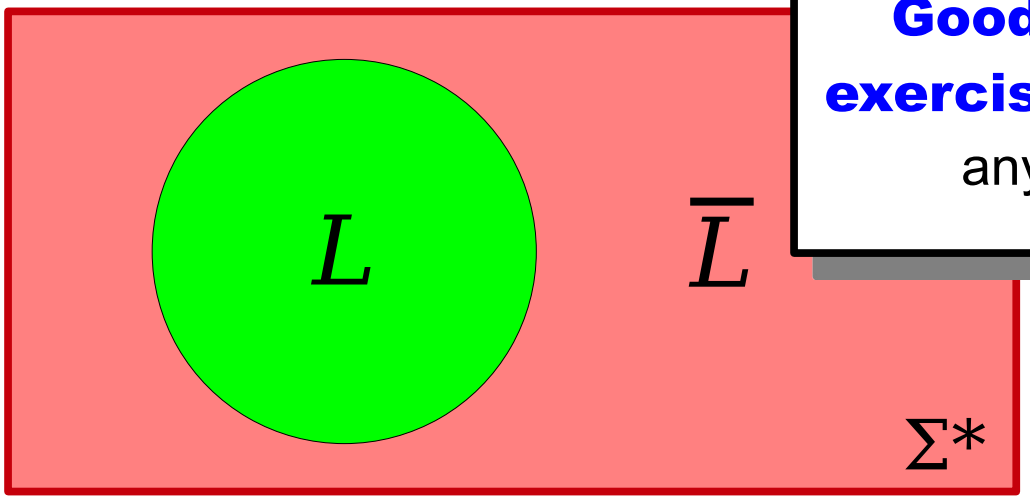
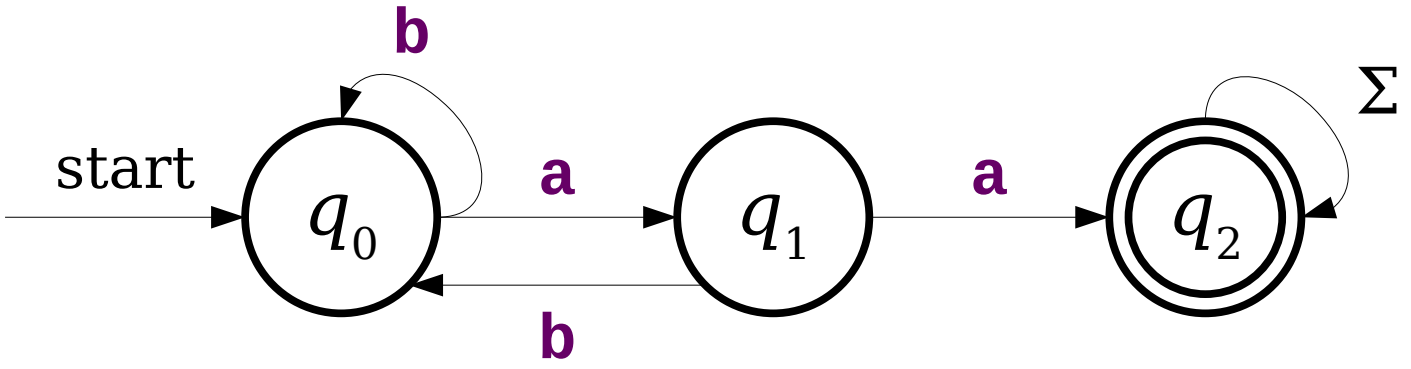


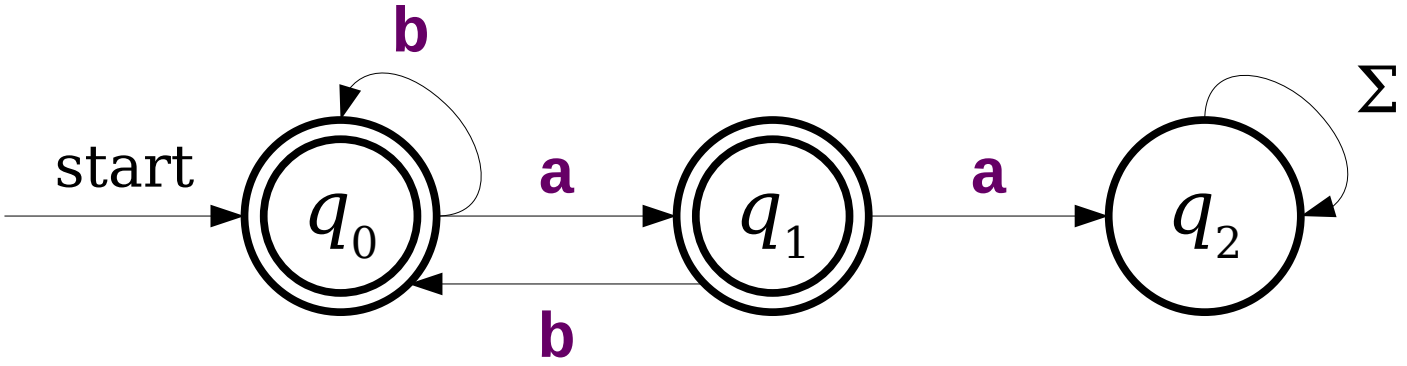
Good proofwriting exercise: prove $\overline{\overline{L}} = L$ for any language L.

Complementing Regular Languages

$L = \{ w \in \{\mathbf{a}, \mathbf{b}\}^* \mid w$ contains **aa** as a substring $\}$

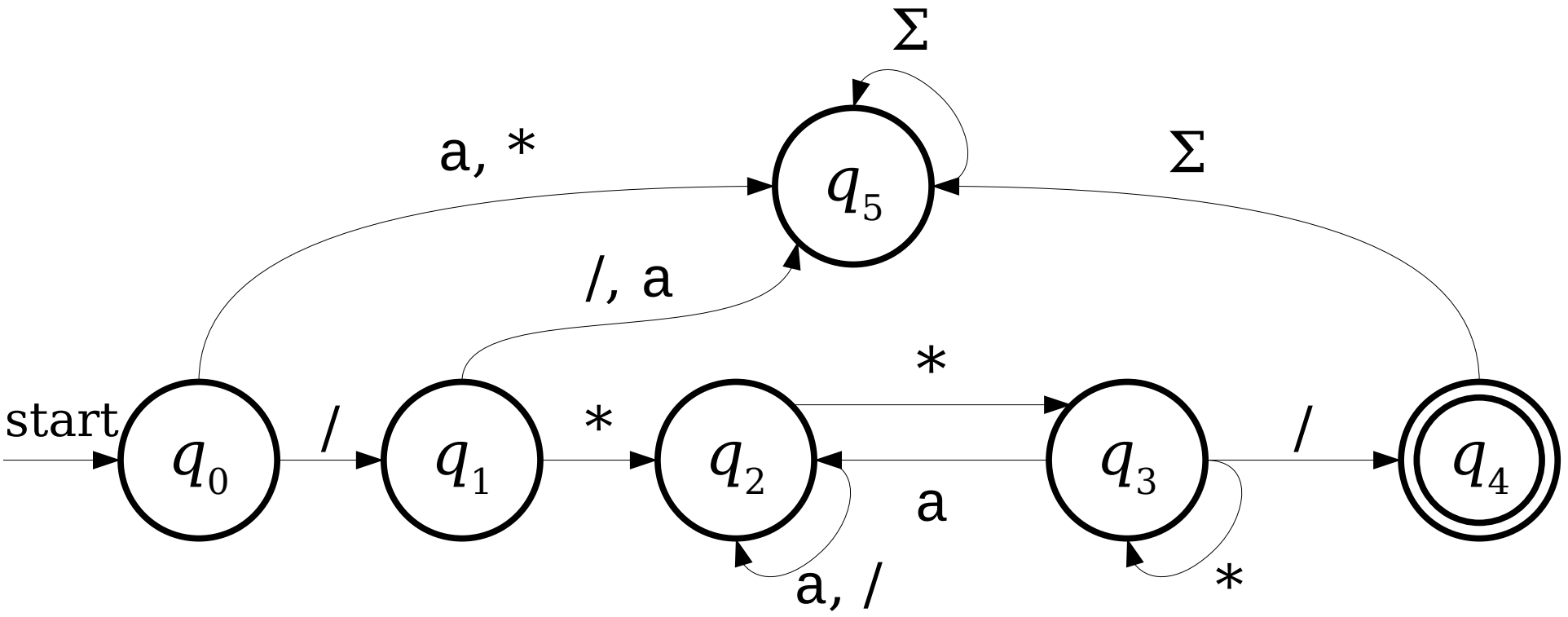


$\overline{L} = \{ w \in \{\mathbf{a}, \mathbf{b}\}^* \mid w$ ***does not*** contain **aa** as a substring $\}$



Complementing Regular Languages

$L = \{\ w \in \{\mathbf{a}, \mathbf{*}, \mathbf{/}\}^* \mid w \text{ represents a C-style comment}\ \}$



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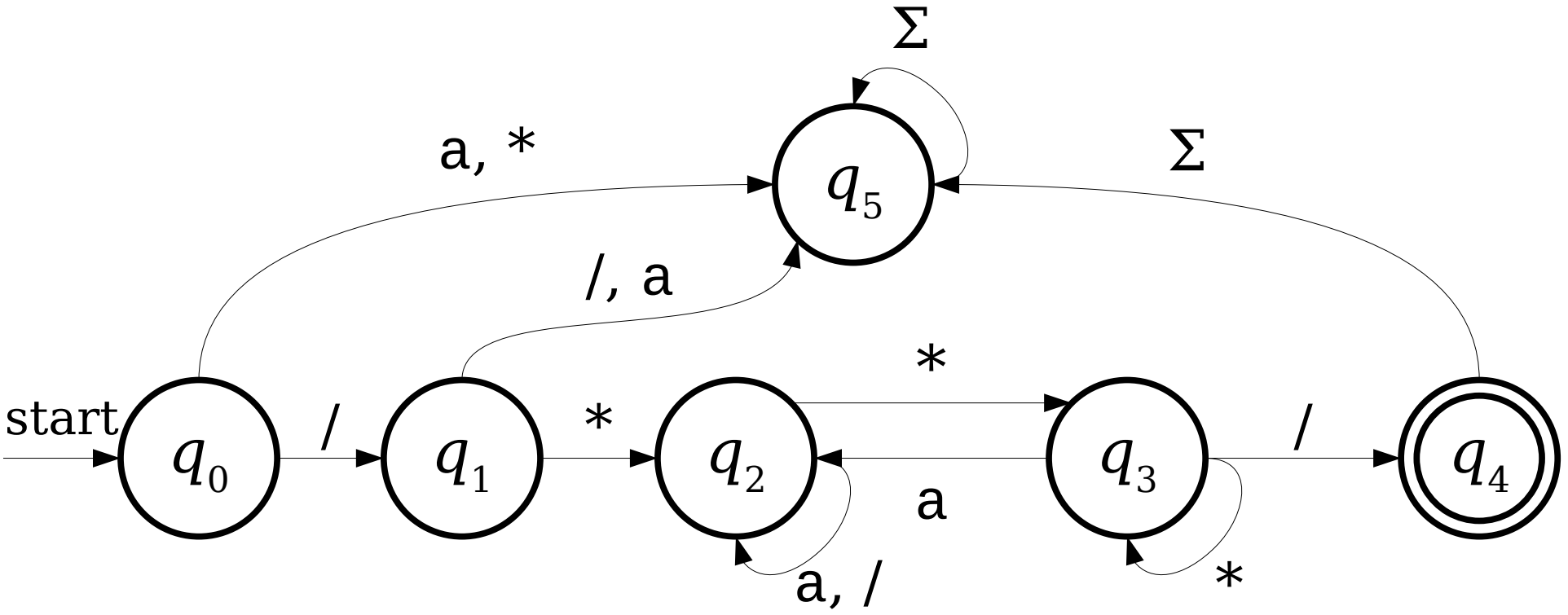

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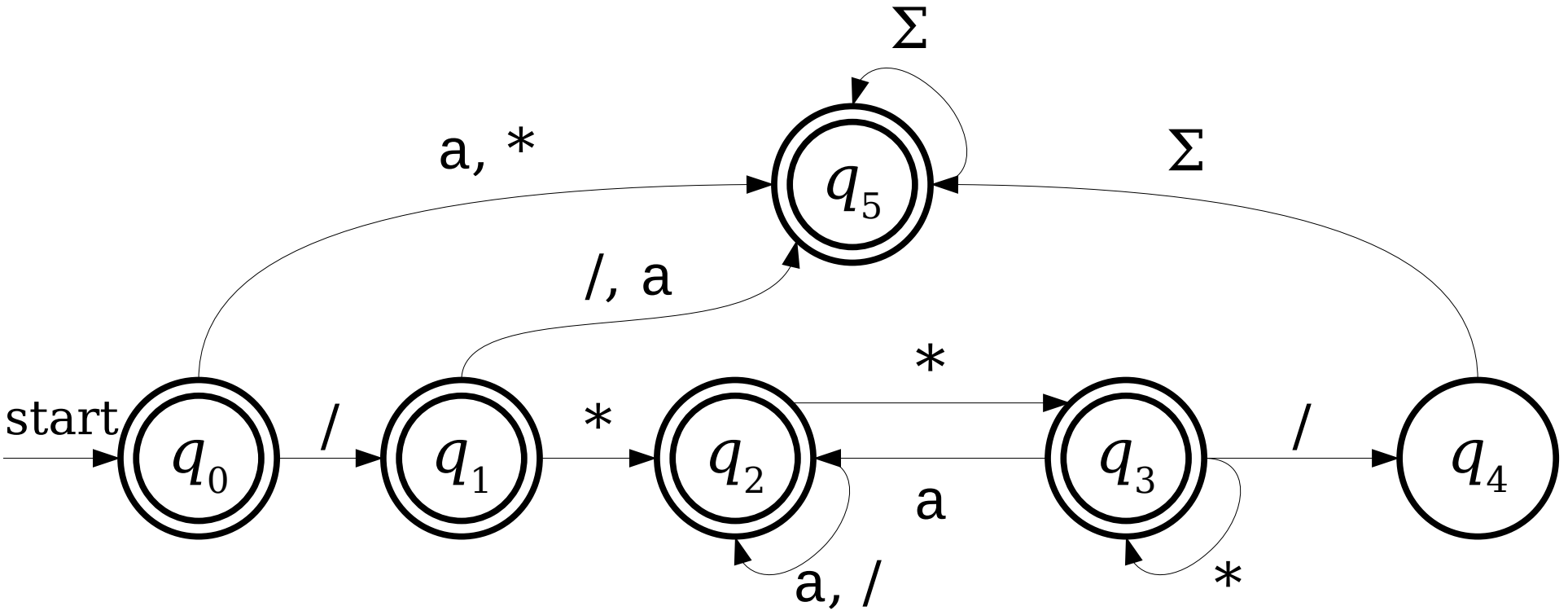
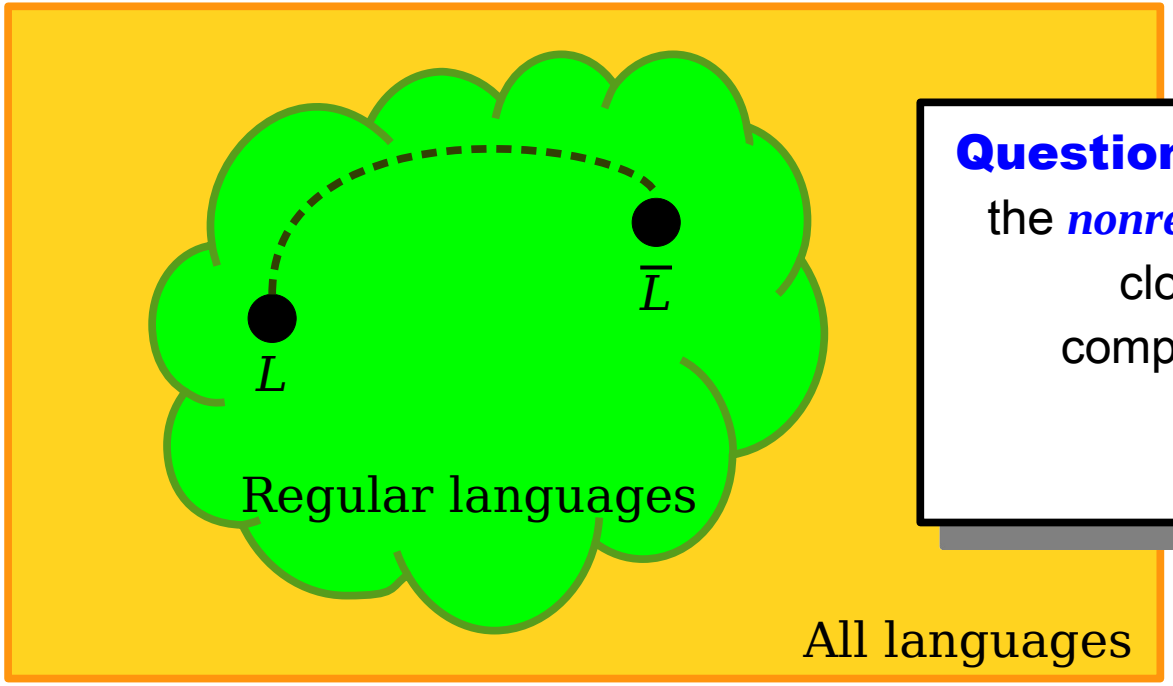


Closure Properties

- ***Theorem:*** If L is a regular language, then $\overline{L}$ is also a regular language.
- As a result, we say that the regular languages are ***closed under complementation***.



Question to ponder: are the ***nonregular*** languages closed under complementation?

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

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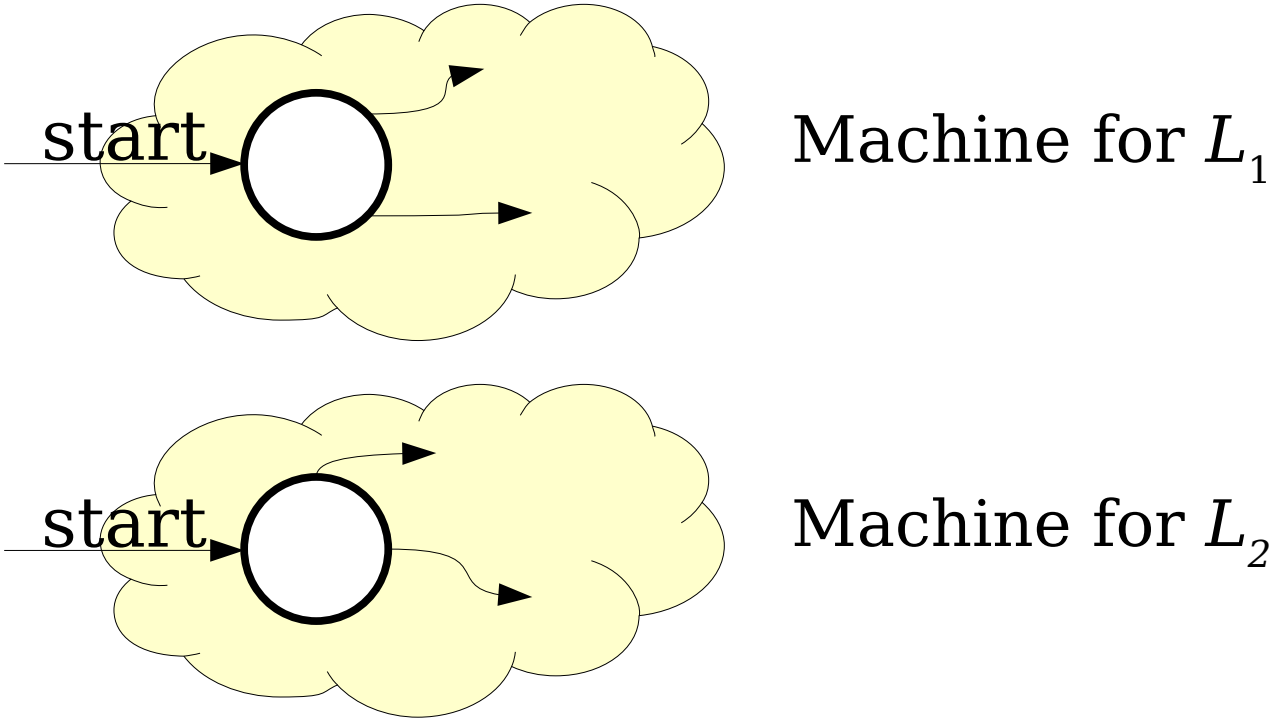

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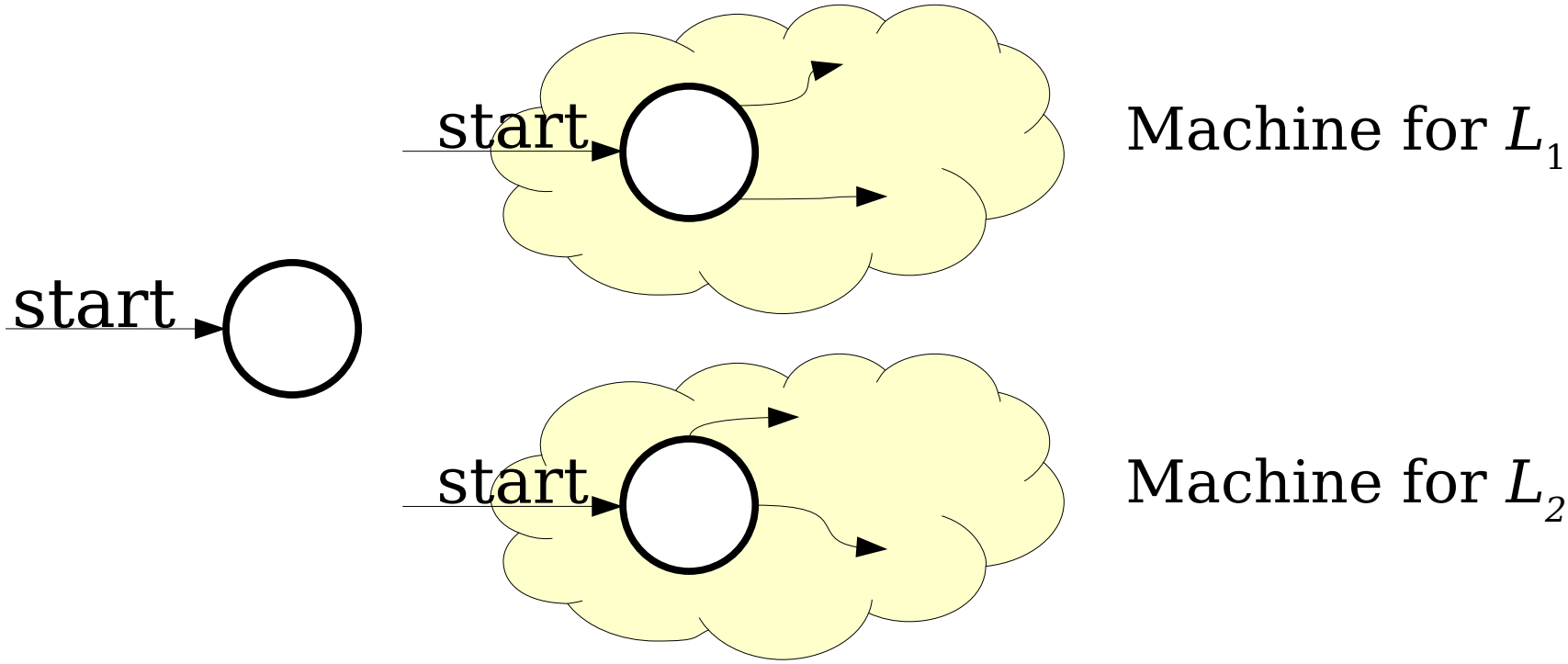

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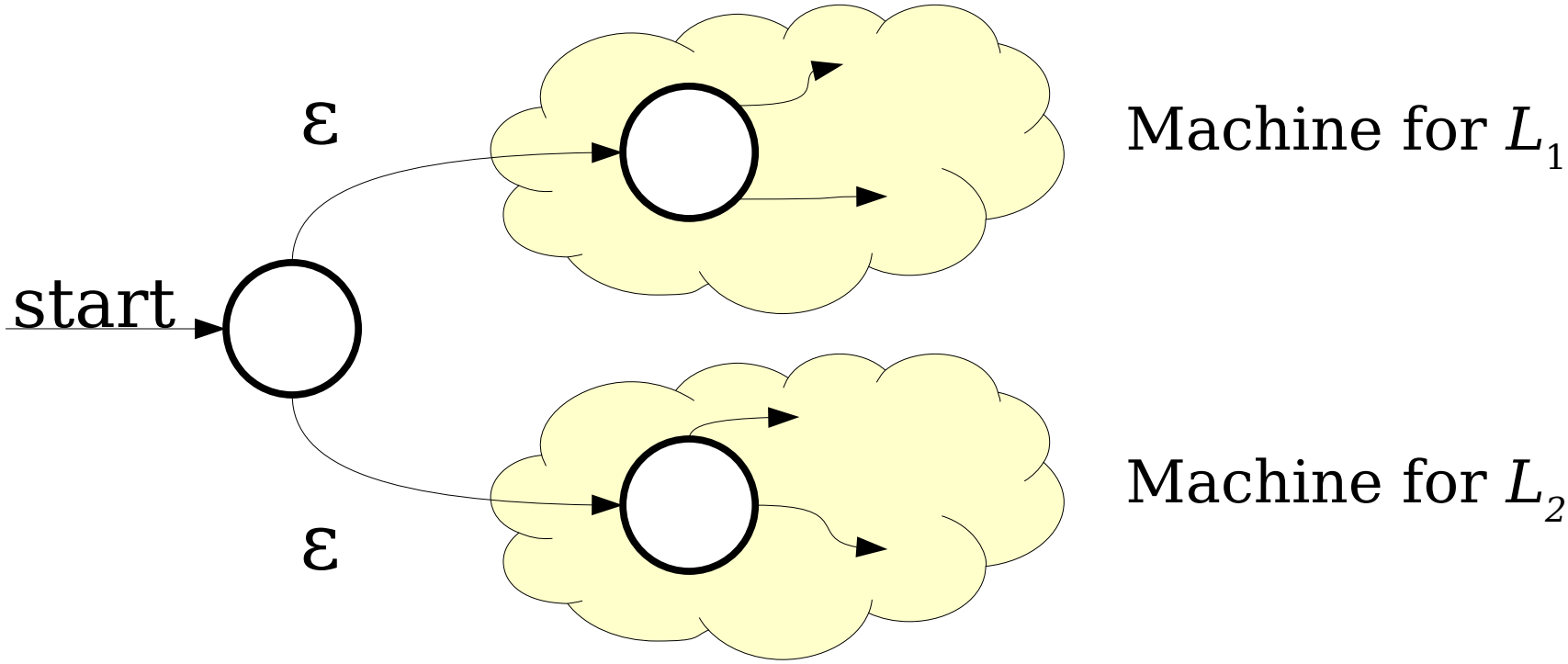

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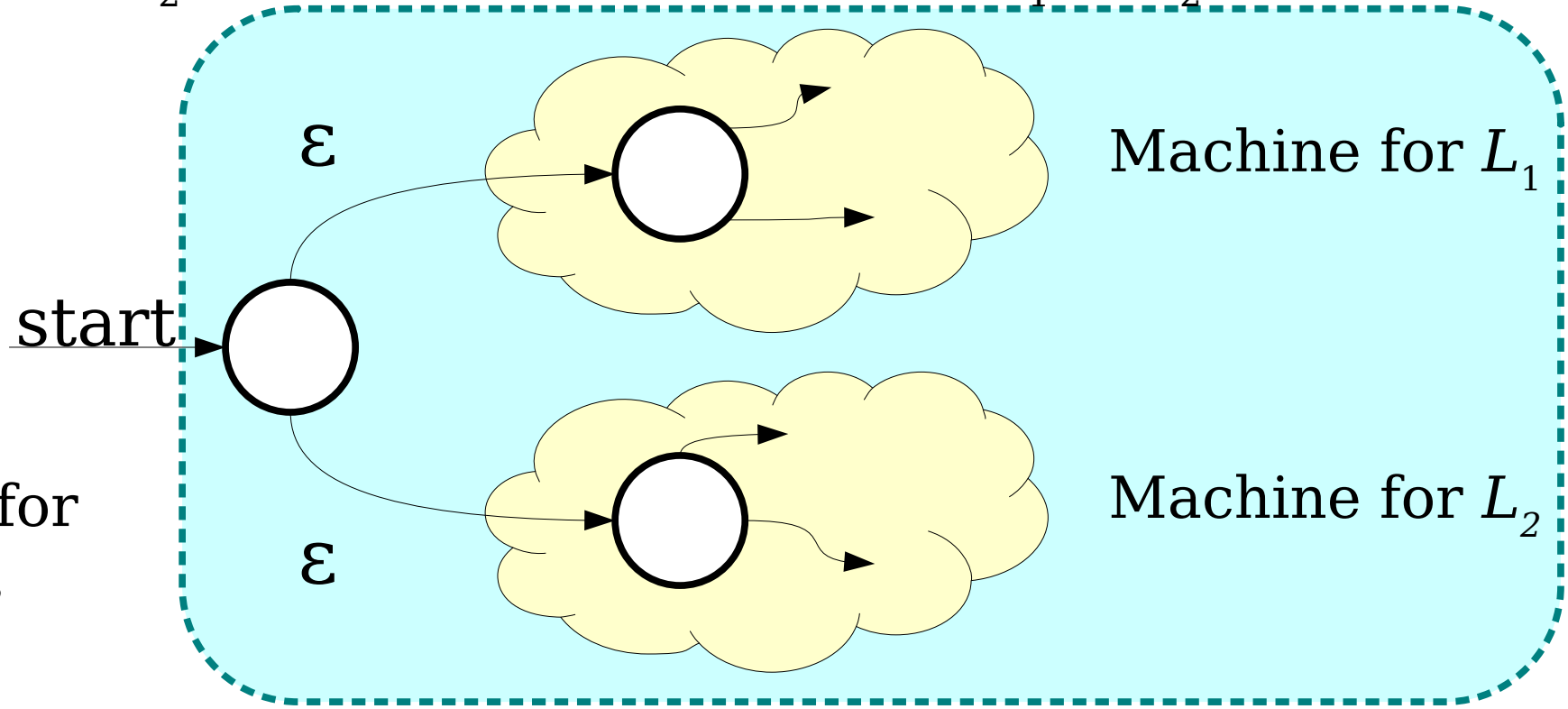

Machine for $L_1 \cup L_2$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?

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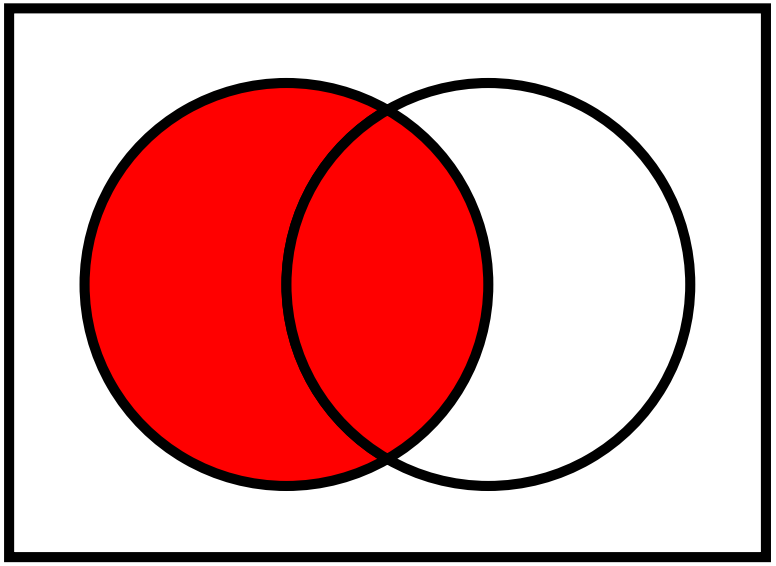
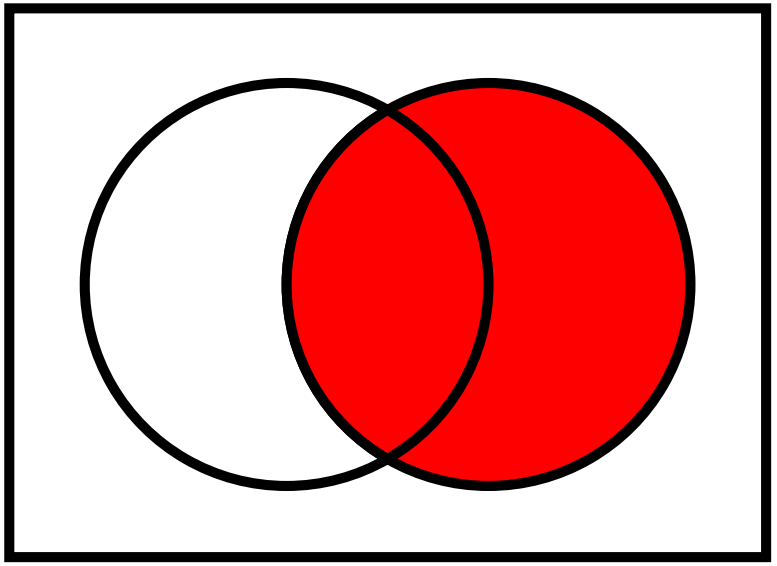


L_1



L_2

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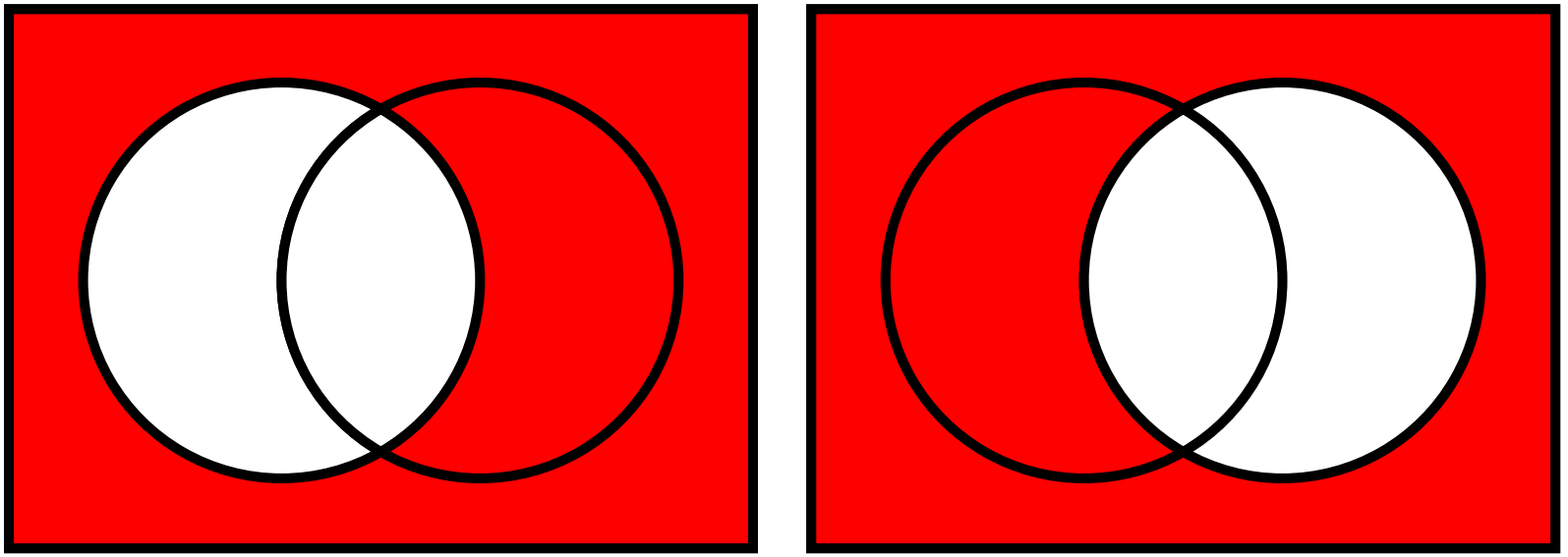

$\overline{L}_1$ $\overline{L}_2$

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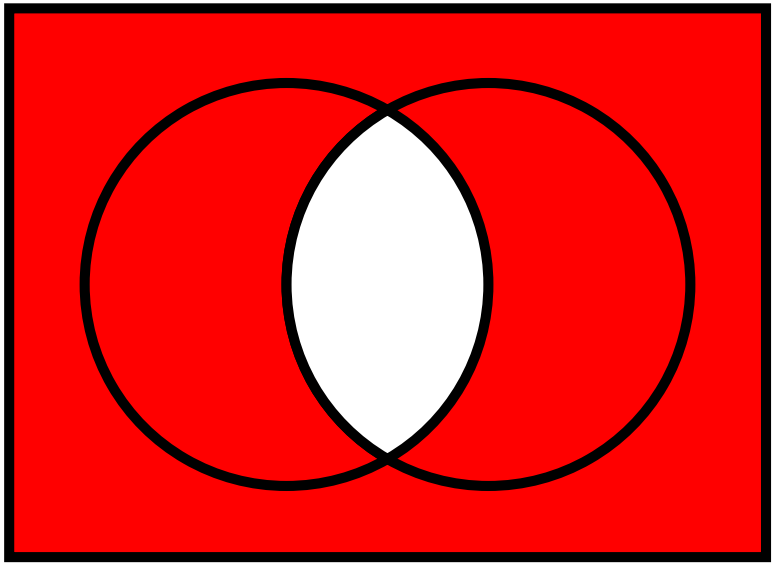

$$\overline{L}_1 \cup \overline{L}_2$$

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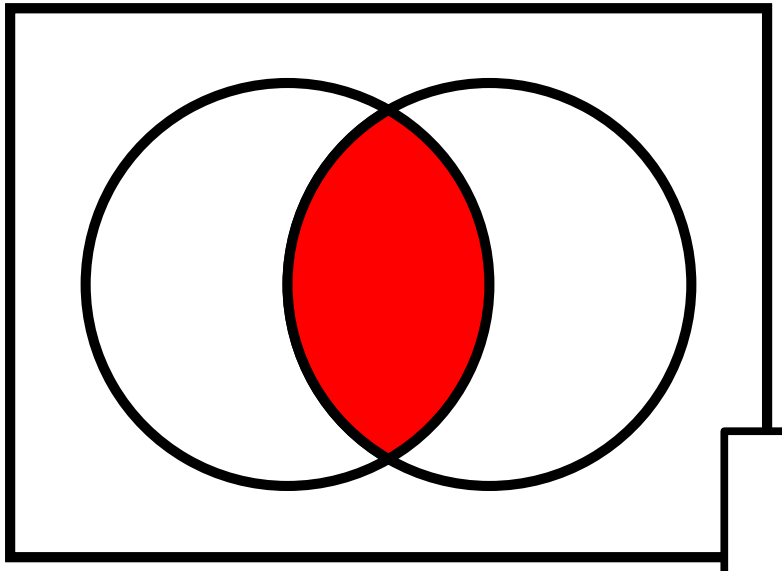

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of w and x, denoted ***wx***, is the string formed by tacking all the characters of x onto the end of w.
- Example: if w = **quo** and x = **kka**, the concatenation wx = **quokka**.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

    $$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

    $$wxy = w(xy) = (wx)y$$

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

Concatenation Example

- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Puppy**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Hugs**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is
  - { **ThePuppyHugsTheWhale**, **TheWhaleLovesTheRainbow**, **TheRainbowJugglesTheRainbow**, … }

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

- Two views of L_1L_2:
  - The set of all strings that can be made by concatenating a string in L_1 with a string in L_2.
  - The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?

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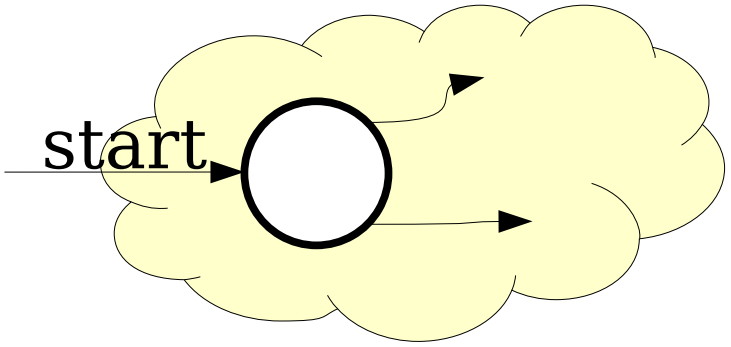
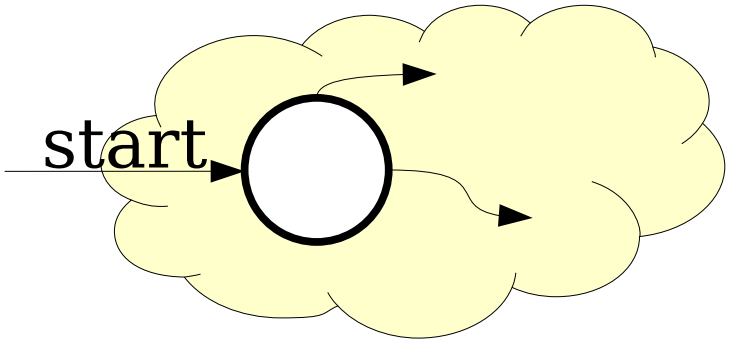


Machine for L_1



Machine for L_2

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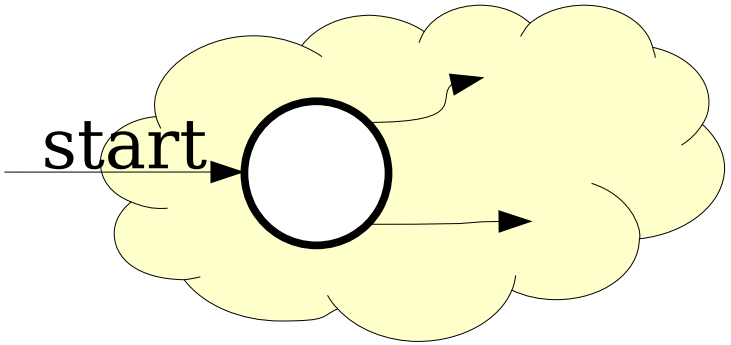

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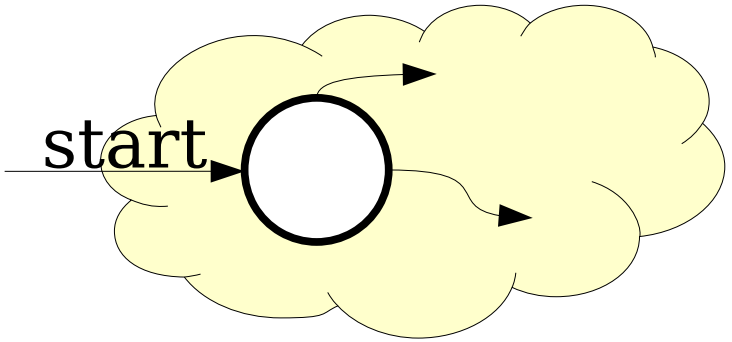

Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

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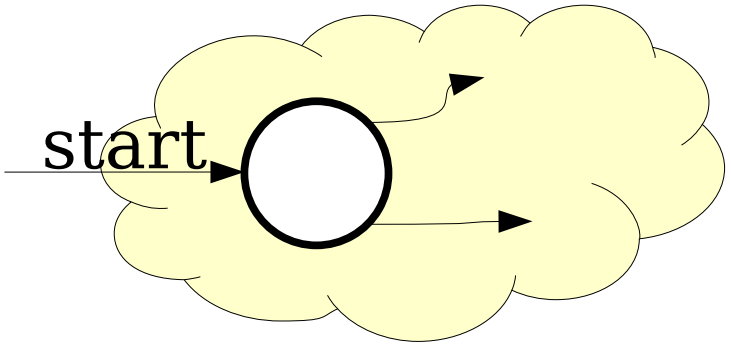

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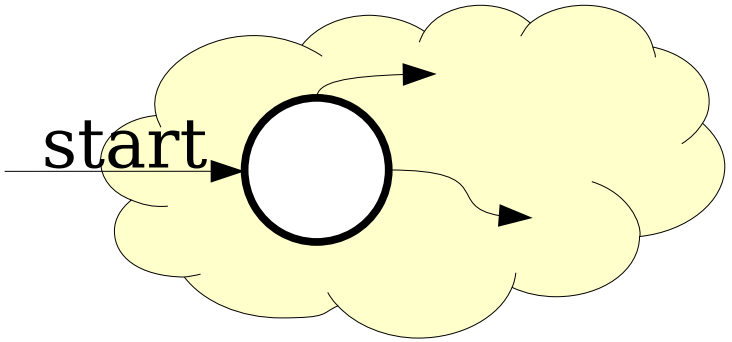

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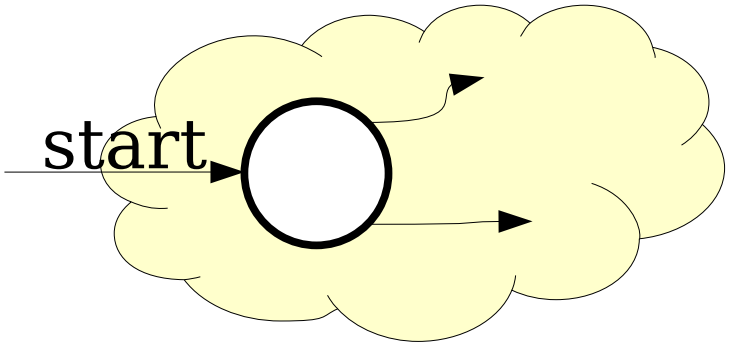

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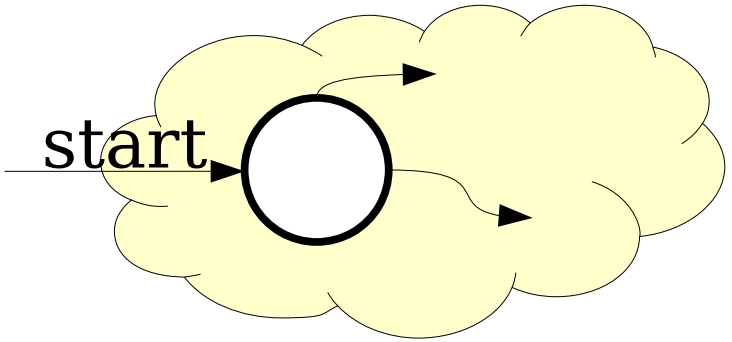

Machine for L_2

b	o	o	k

k	e	e	p	e	r

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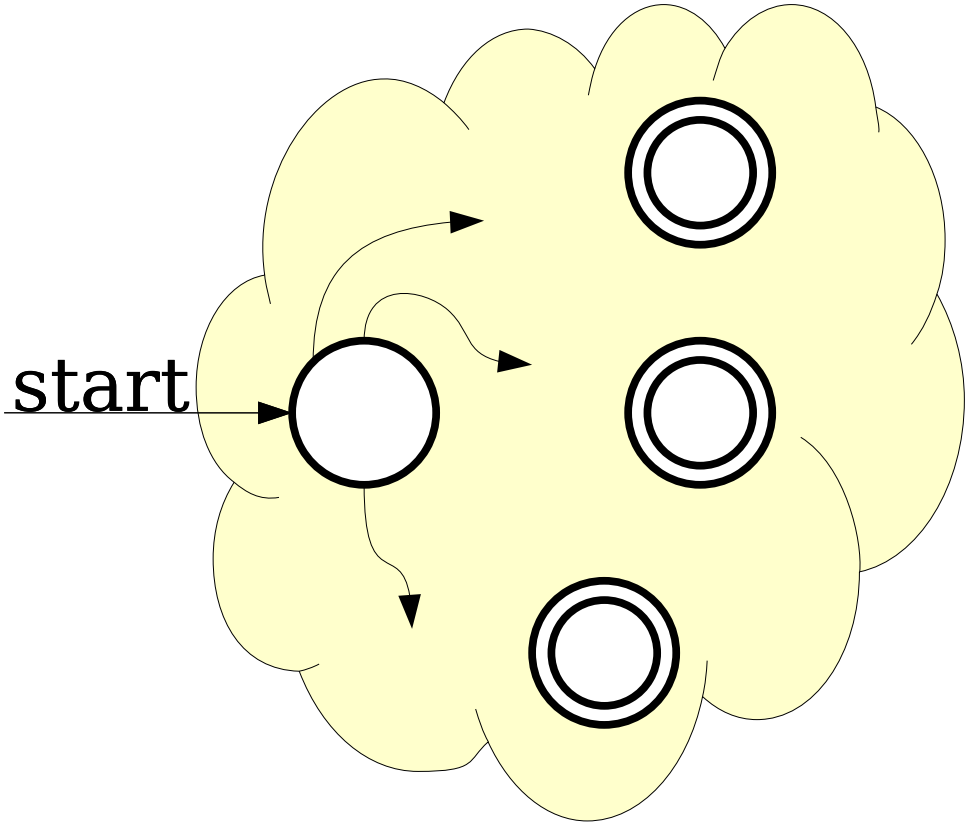
- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea***:
  - Run a DFA/NFA for L_1 on w.
  - Whenever it reaches an accepting state, optionally hand the rest of w to a DFA/NFA for L_2.
  - If the automaton for L_2 accepts the rest, $w \in L_1L_2$.
  - If the automaton for L_2 rejects the remainder, the split was incorrect.

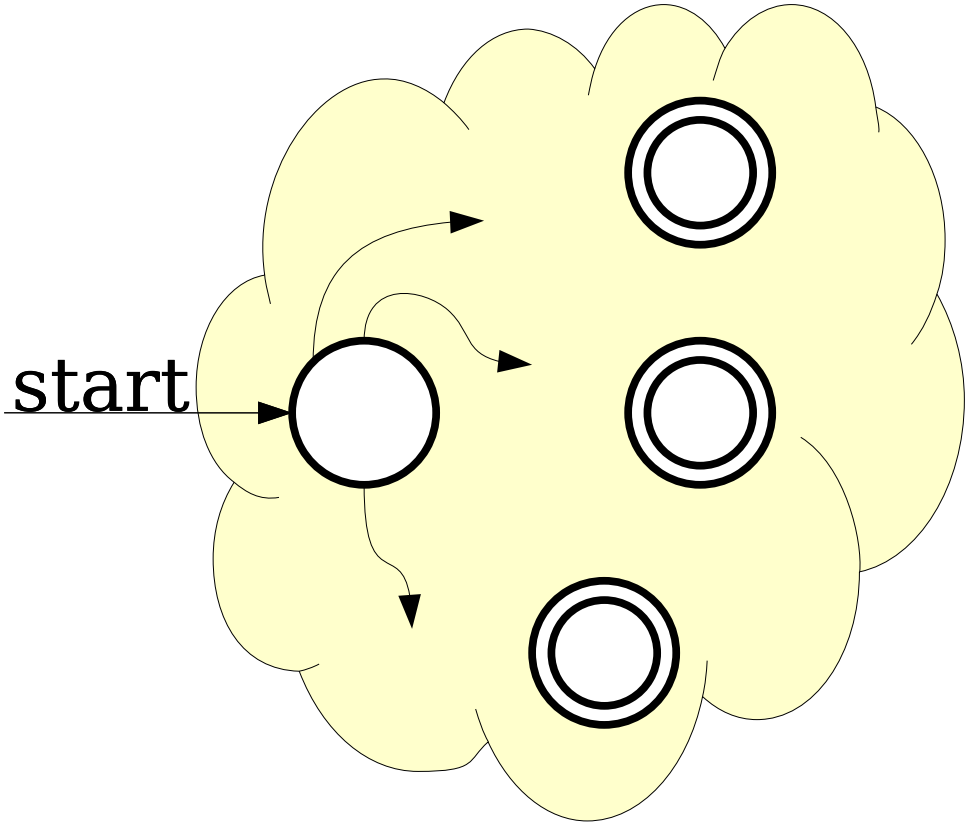
Concatenating Regular Languages

Concatenating Regular Languages



Machine for L_1

Concatenating Regular Languages



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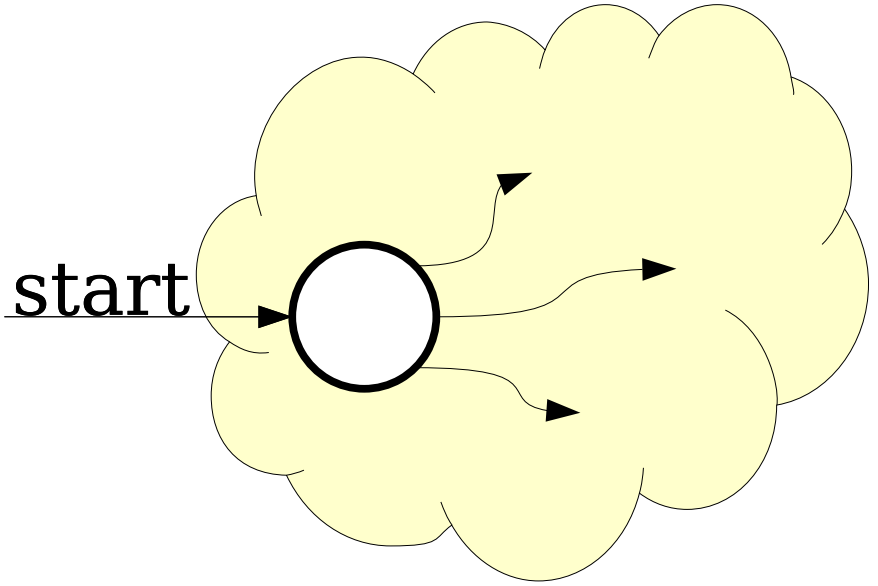
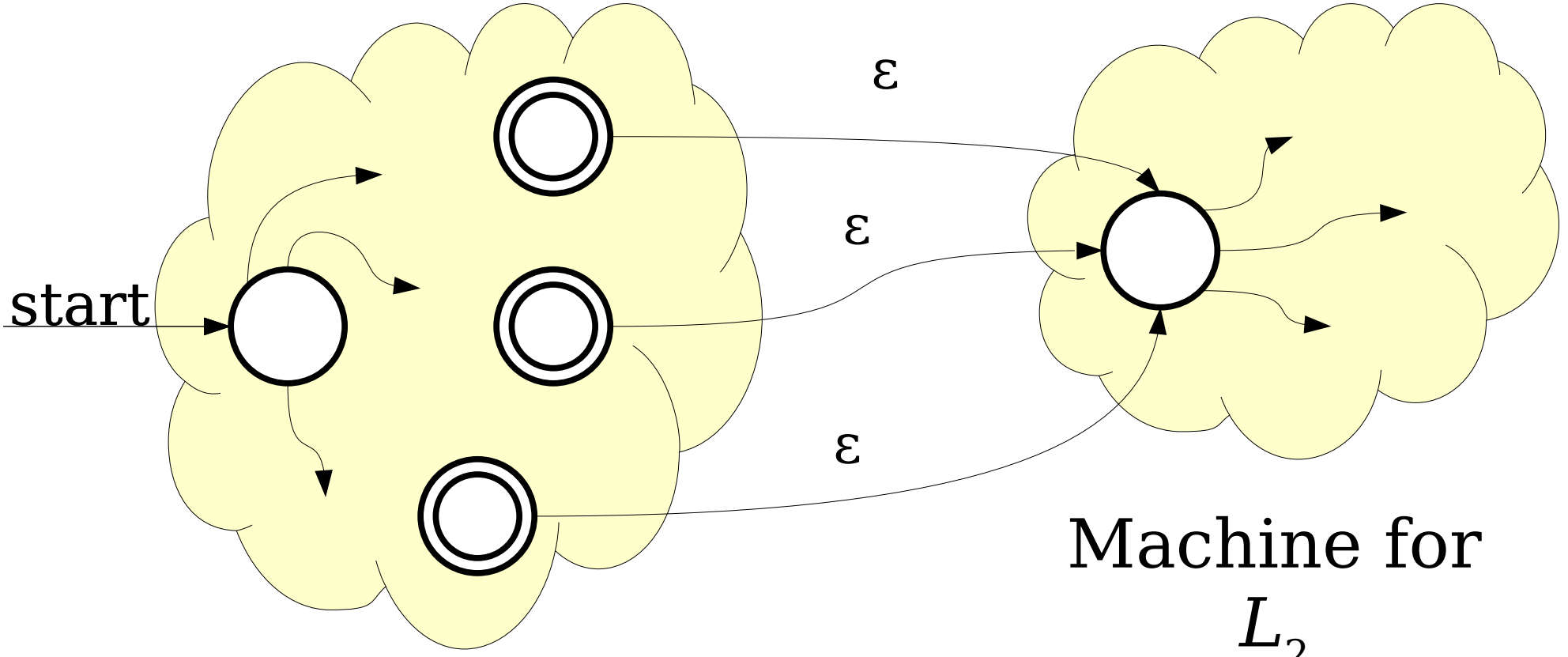


Machine for L_2

Concatenating Regular Languages



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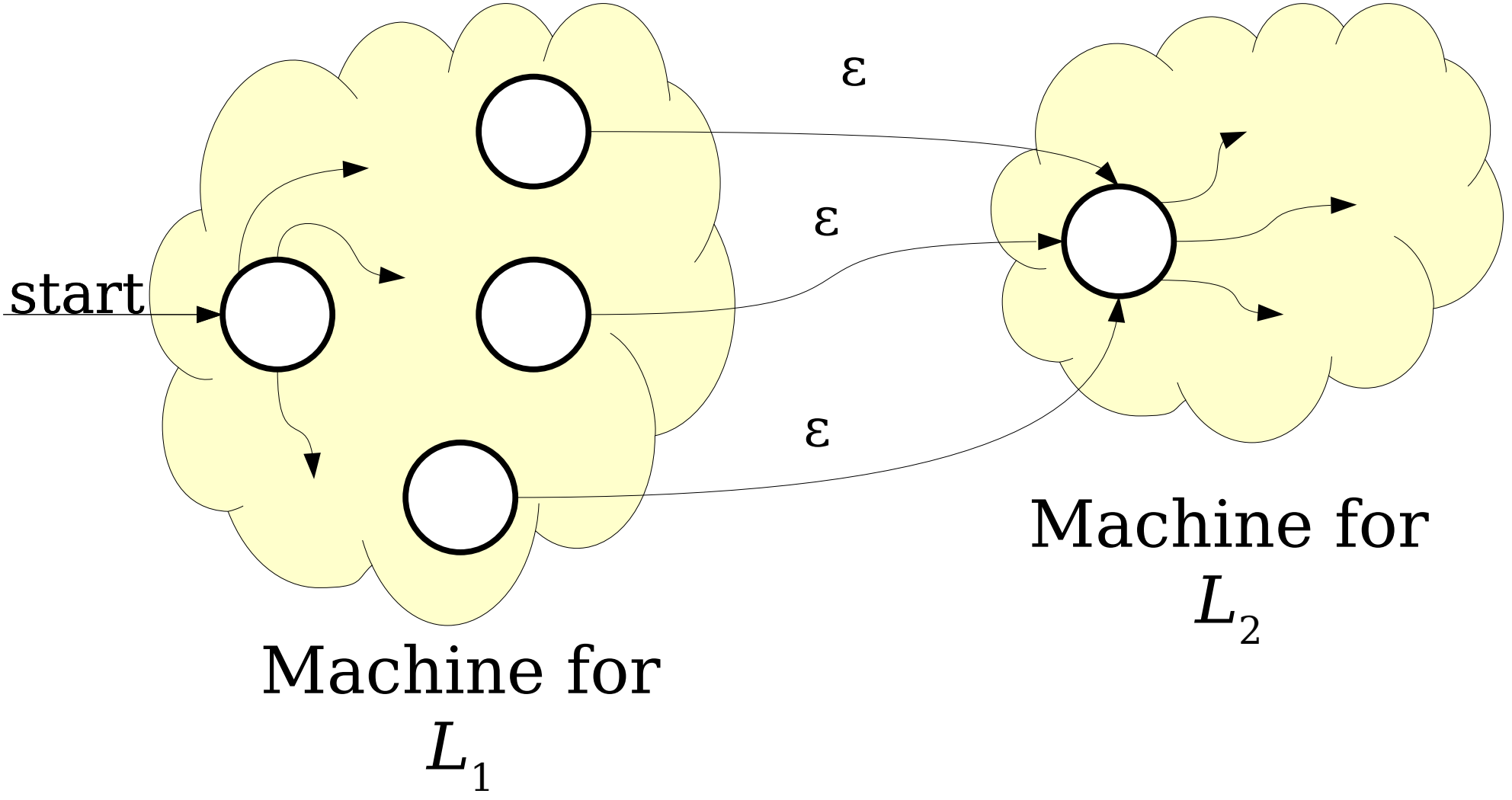

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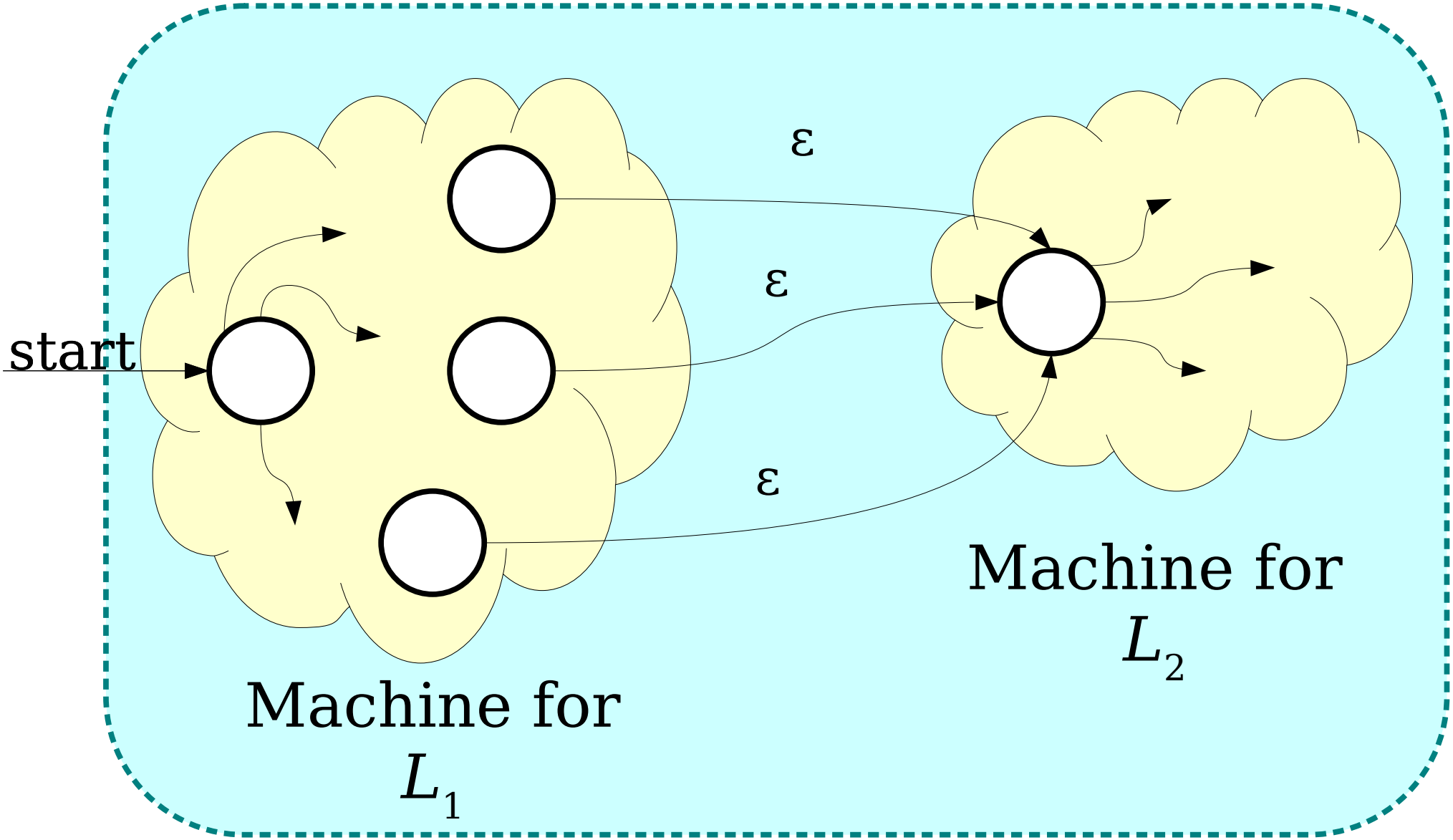


Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \varnothing$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\varnothing^0$?

The Kleene Star

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, L^* is the language all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question to ponder:*** What is Ø*?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

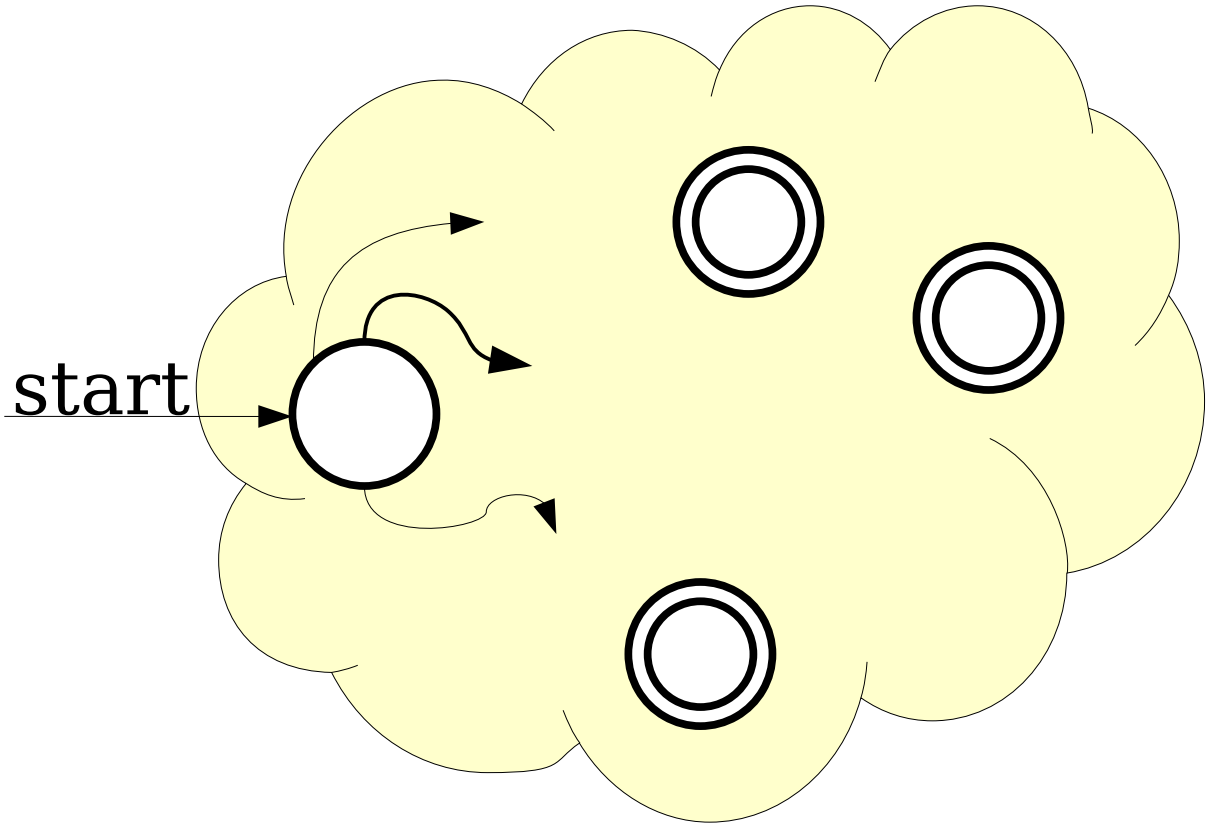
…

}

Think of L^* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

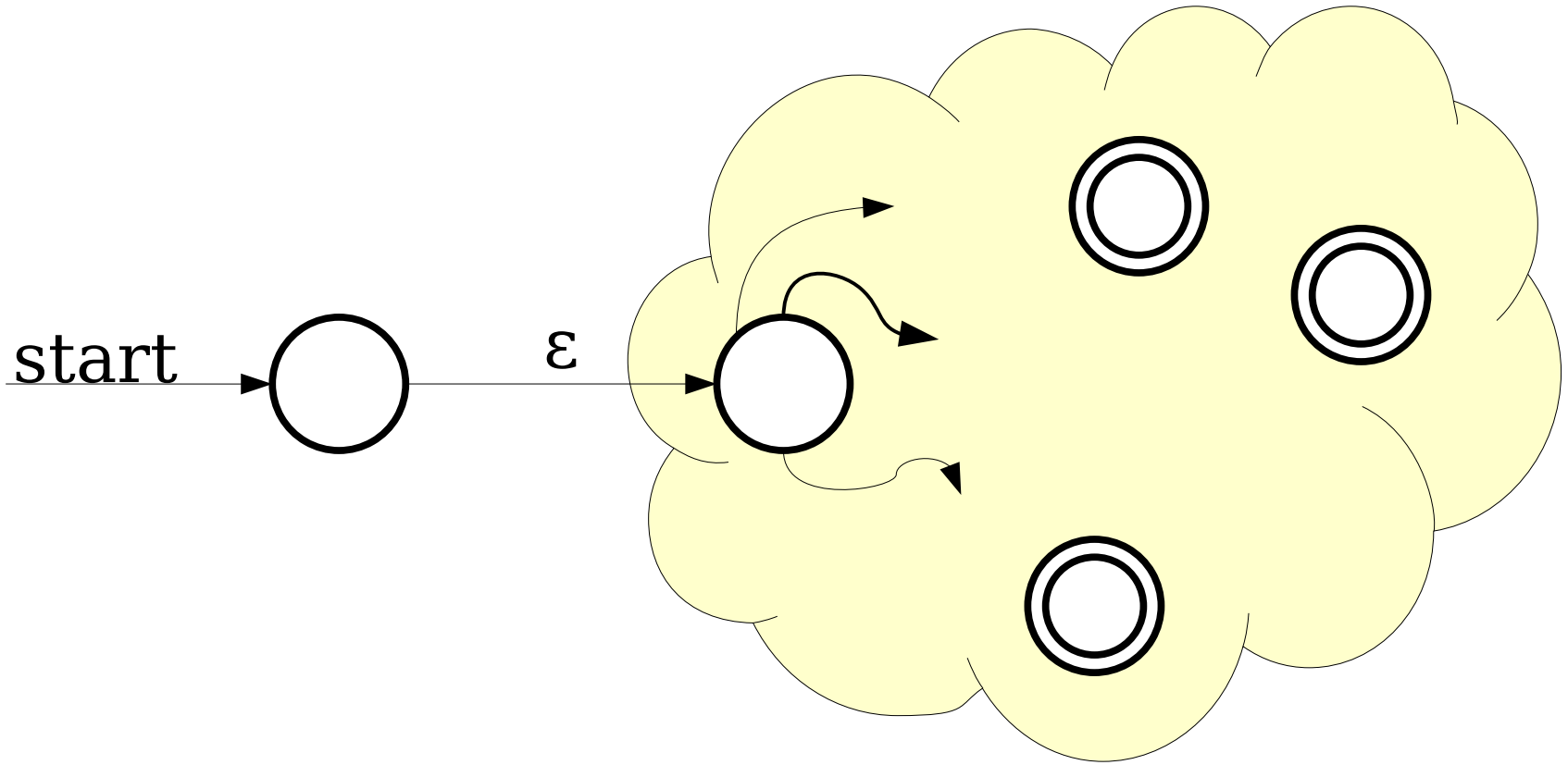
Idea: Can we convert an NFA for language L to an NFA for language L*?

The Kleene Star



Machine for L

The Kleene Star



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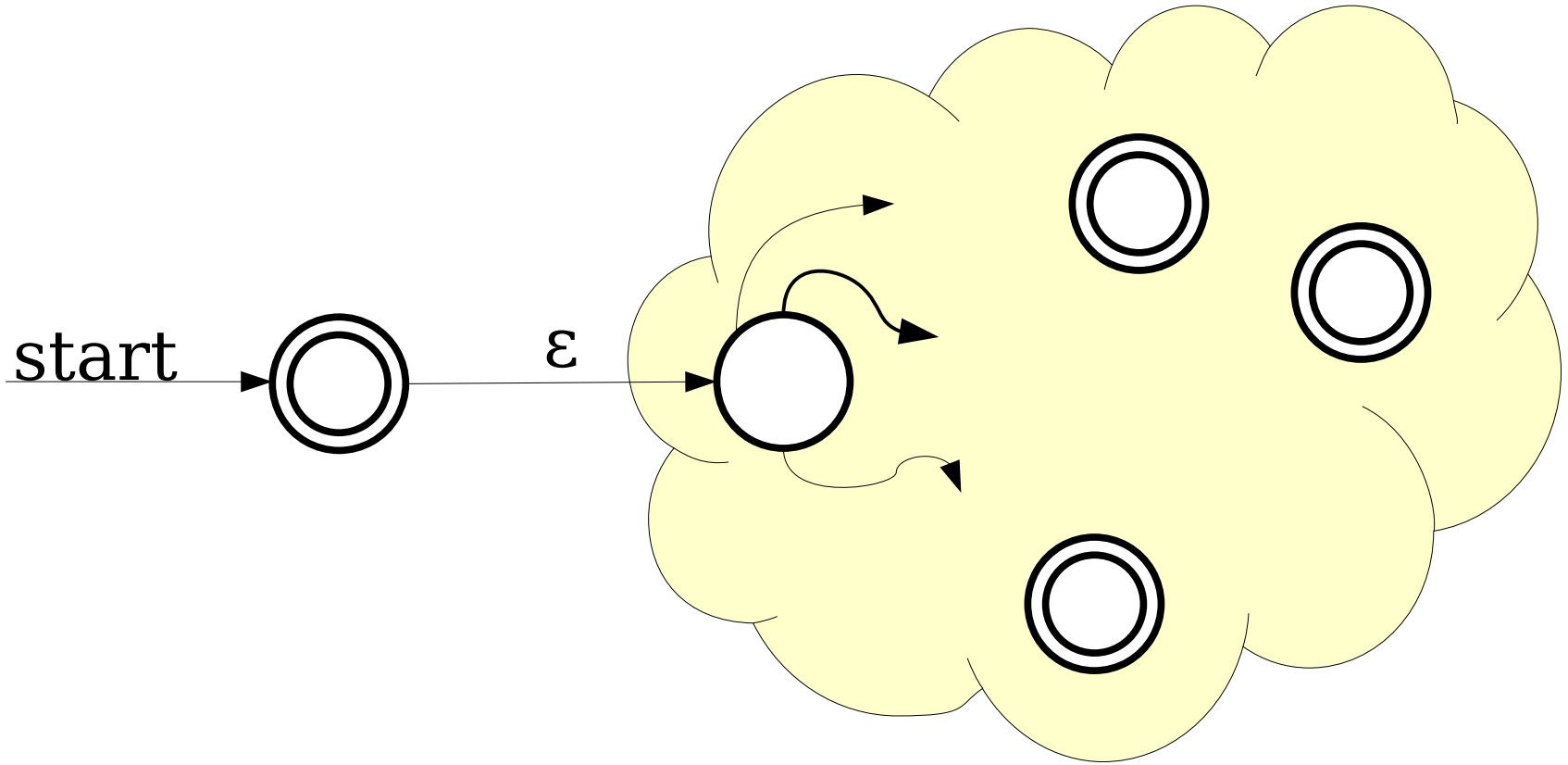
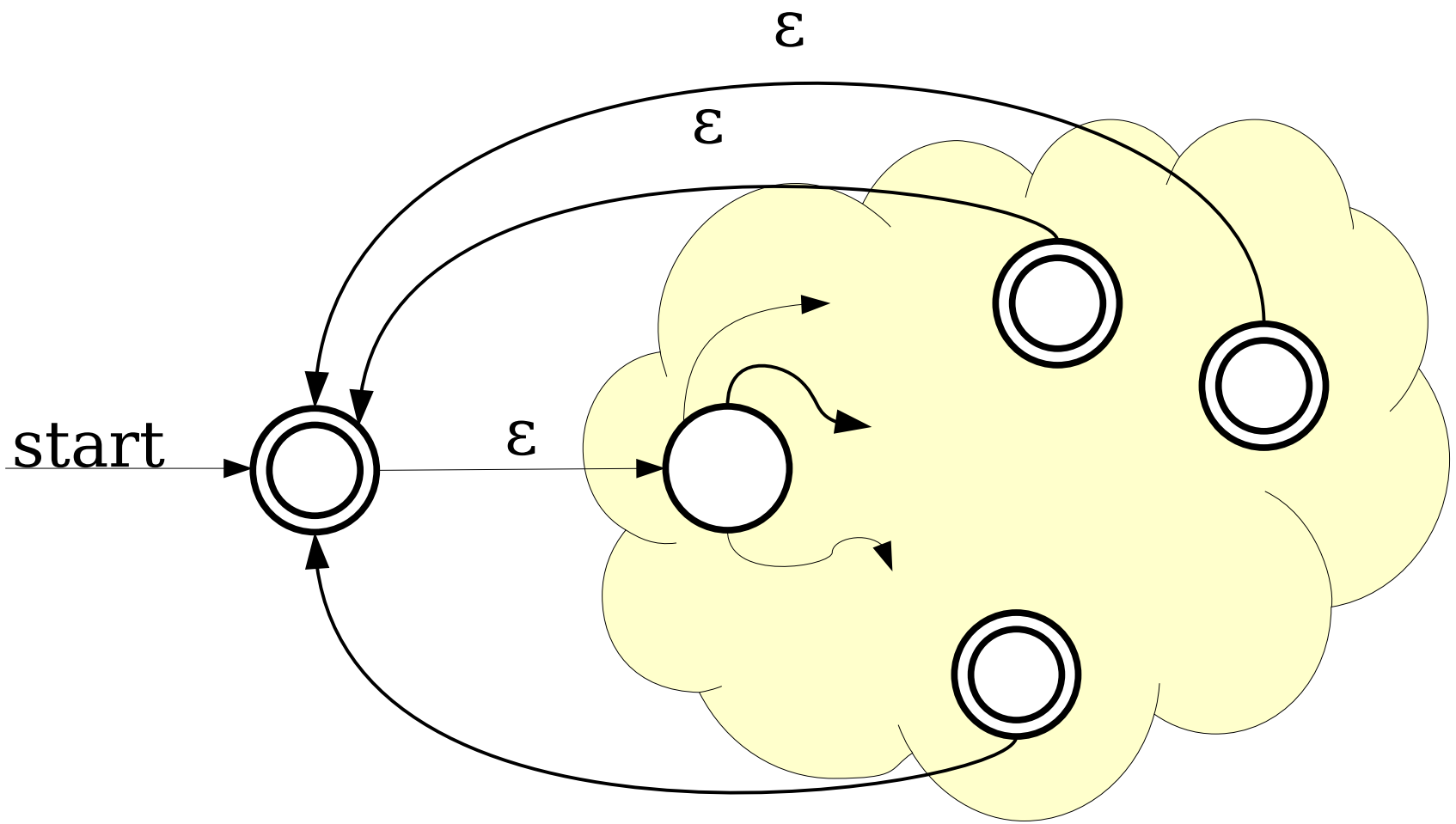
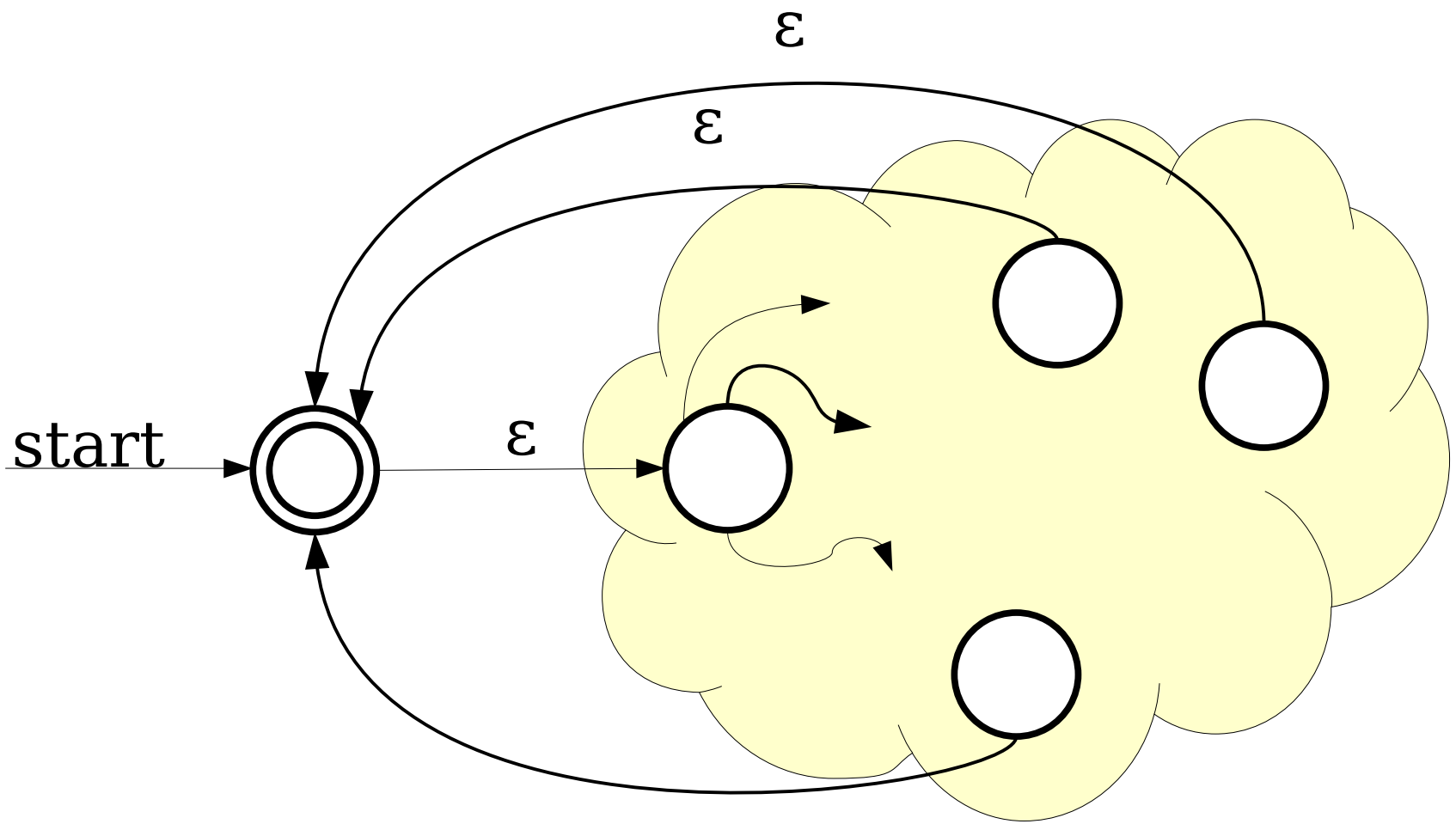


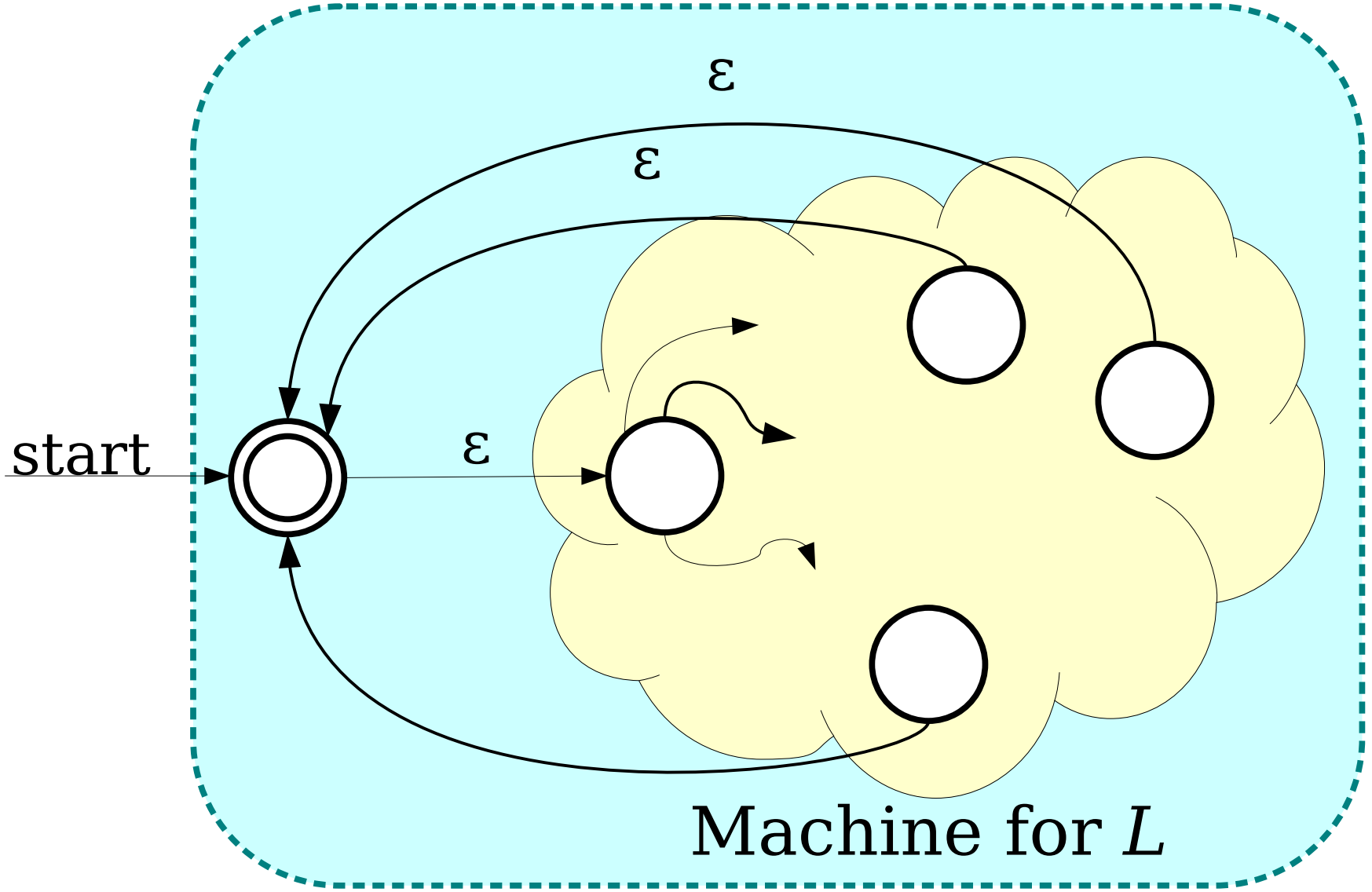
The Kleene Star



The Kleene Star

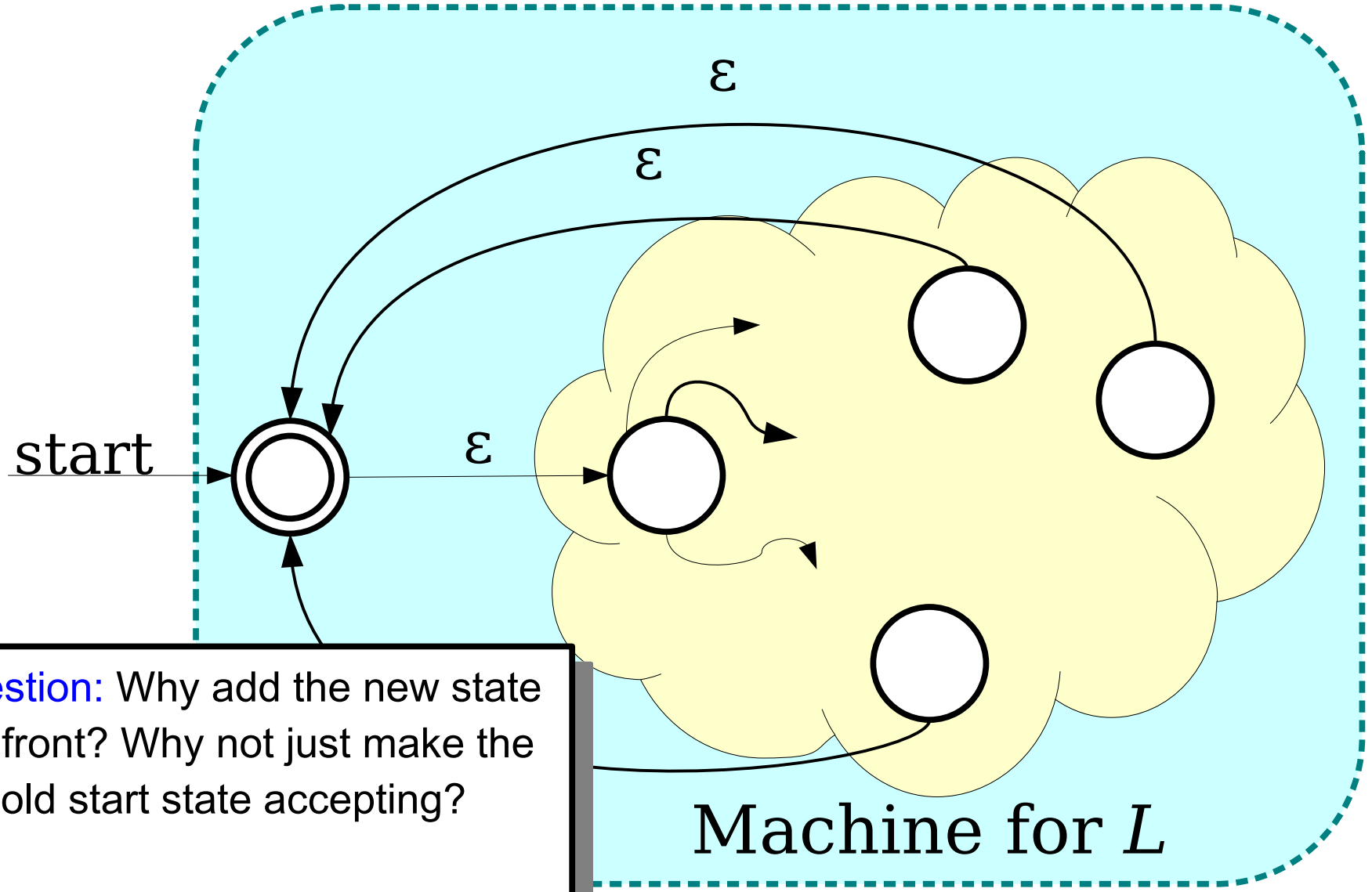


The Kleene Star



Machine for L^*

The Kleene Star



Question: Why add the new state out front? Why not just make the old start state accepting?

achine for L^*

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L}_1$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!